GEOMETRY OF STANDARD SYMMETRIZED TENSORS

HANK G. HARMON   RANDALL R. HOLMES

Abstract. The geometric properties of the set of standard (decomposable) symmetrized tensors are studied and some general results are obtained. As an example, the geometry is worked out completely in the case where the group is a dihedral group, and this result is used to give a more conceptual proof of an earlier result. As another example, it is shown that there exists an orbital subspace such that the standard symmetrized tensors in the subspace form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced.

0. Introduction

Let G be a subgroup of the symmetric group S_n ($n \in \mathbf{N}$) and let V be an inner product space. Orthogonality properties of the set of standard (decomposable) symmetrized tensors in $V^{\otimes n}$ corresponding to G have been studied for more than two decades [WG91, HT92, Hol95, DP99, BPR03, Hol04, TS12]. The determination of such properties would be facilitated by an understanding of the more general geometric properties of this set. We propose a framework for the study of such properties.

The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces, so it is sufficient to study the sets of standard symmetrized tensors in these subspaces. It then follows that it is sufficient to study for each irreducible character χ of G and each subgroup H of G the set Ψ of standard vectors in the coset space $\mathcal{C}_H^\chi$ (see Section 3).

In Section 4 we obtain some general results about the pairs $(\mathcal{C}_H^\chi, \Psi)$. Then in Section 5 we compute all such pairs in the case where G is a dihedral group and use our results to give a more conceptual proof of an earlier result in [Hol04]. Finally, in Section 6 we generalize a result of Torres and Silva [TS12] by showing that there exists an orbital subspace such that the standard symmetrized tensors in the subspace form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced.

2010 *Mathematics Subject Classification.* 15A69, 20C15, 20C30.

Key words and phrases. Symmetrized tensors, coset space, dihedral group, root system.

1. Hermitian form

In this section and the next we review, for the convenience of the reader, some standard (and also some less standard) terminology and results.

Let V be a complex vector space. A function $f : V \times V \to \mathbf{C}$ is a *Hermitian form on* V if for all $u, v, w \in V$ and $\alpha \in \mathbf{C}$ the following hold:

(i) $f(u+v, w) = f(u, w) + f(v, w)$,
(ii) $f(\alpha v, w) = \alpha f(v, w)$,
(iii) $f(v, w) = \overline{f(w, v)}$.

Let f be a Hermitian form on V. It follows from the axioms that f is antilinear in the second argument (meaning $f(u, v+w) = f(u, v) + f(u, w)$ and $f(v, \alpha w) = \bar{\alpha} f(v, w)$ for all $u, v, w \in V$ and $\alpha \in \mathbf{C}$) and that $f(v, v) \in \mathbf{R}$ for all $v \in V$.

The Hermitian form f is *positive semidefinite* if $f(v, v) \geq 0$ for all $v \in V$; it is an *inner product* if it is positive semidefinite and it satisfies the *definite* property: $f(v, v) = 0$ if and only if $v = 0$.

The *kernel* of f is the subspace $\ker f = \{v \in V \mid f(v, w) = 0 \text{ for all } w \in V\}$ of V. Put $\bar{V} = V/\ker f$ and denote by $v \mapsto \bar{v}$ the canonical epimorphism $V \to \bar{V}$. (Context should keep any confusion from arising between this notation and that for complex conjugation.) The function $\bar{f} : \bar{V} \times \bar{V} \to \mathbf{C}$ given by $\bar{f}(\bar{v}, \bar{w}) = f(v, w)$ is a well-defined Hermitian form on $\bar{V}$.

1.1 Lemma. *Let f be a positive semidefinite Hermitian form on V.*

(i) $\ker f = \{v \in V \mid f(v, v) = 0\}$.
(ii) *The function $\bar{f}$ is an inner product on $\bar{V}$.*

Proof. (i) Let $v \in \{v \in V \mid f(v, v) = 0\} =: W$. Then $\|v\| = f(v, v)^{1/2} = 0$, so $|f(v, w)| \leq \|v\|\|w\| = 0$ for all $w \in V$, where we have used the Cauchy-Schwartz inequality (the proof of which does not require the definite property). Therefore, $v \in \ker f$ and we conclude that $W \subseteq \ker f$. The other inclusion follows immediately from the definition of $\ker f$.

(ii) The Hermitian form $\bar{f}$ is positive semidefinite, so it is enough to show that it satisfies the definite property. Let $v \in V$ and assume that $\bar{f}(\bar{v}, \bar{v}) = 0$. Then $f(v, v) = 0$ so that $v \in \ker f$ by (i). Therefore $\bar{v} = 0$ as desired. □

2. Similarity transformation

Let V be an inner product space (i.e., a complex vector space with an inner product, which we denote by $(\,\cdot\,,\,\cdot\,)$). The inner product on V induces a norm on V given by $\|v\| = (v, v)^{1/2}$.

2.1 Proposition. *Let V and V' be inner product spaces, let $\varphi : V \to V'$ be a linear map, and let r be a positive real number. The following are equivalent:*

(i) $\|\varphi(v)\| = r\|v\|$ *for all* $v \in V$.
(ii) $(\varphi(v), \varphi(w)) = r^2(v, w)$ *for all* $v, w \in V$.

Proof. Assume that (i) holds. For v and w in either V or V', the polarization identity holds:

$$(v,w) = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2\right).$$

Therefore, using the linearity of φ and (i), we get $(\varphi(v), \varphi(w)) = r^2(v,w)$ for all $v, w \in V$, so (ii) holds.

Now assume that (ii) holds. For every $v \in V$, we have

$$\|\varphi(v)\| = (\varphi(v), \varphi(v))^{1/2} = r(v,v)^{1/2} = r\|v\|,$$

so (i) holds. □

A linear map $\varphi : V \to V'$ satisfying the equivalent conditions of Proposition 2.1 is a *similarity transformation* (*of ratio* r). Such a map preserves angles as well as relative lengths (i.e., $\|\varphi(v)\|/\|\varphi(w)\| = \|v\|/\|w\|$). A similarity transformation of ratio 1 is an *isometry.*

2.2 Corollary. *Let r be a positive real number. A linear map $\varphi : V \to V'$ is a similarity transformation of ratio r if and only if $\varphi = \psi\mu_r$ for some isometry $\psi : V \to V'$, where $\mu_r : V \to V$ is the homothety given by $\mu_r(v) = rv$.*

Proof. Let $\varphi : V \to V'$ be a linear map. Assume that φ is a similarity transformation of ratio r. The map $\psi : V \to V'$ given by $\psi = \varphi\mu_{1/r}$ is linear and $\varphi = \psi\mu_r$. Moreover, $\|\psi(v)\| = \|\varphi((1/r)v)\| = r\|(1/r)v\| = \|v\|$ for all $v \in V$, so ψ is an isometry. The converse is proved similarly. □

2.3 Proposition. *Let V be a complex vector space with positive semidefinite Hermitian form f, let V' be an inner product space, let r be a positive real number, and let $\varphi : V \to V'$ be a linear map satisfying $(\varphi(v), \varphi(w)) = rf(v,w)$ for all $v, w \in V$. The map $\bar{\varphi} : \bar{V} \to V'$ given by $\bar{\varphi}(\bar{v}) = \varphi(v)$ is a well-defined injective similarity transformation of ratio $\sqrt{r}$.*

Proof. For $v \in V$, we have $(\varphi(v), \varphi(v)) = rf(v,v)$, so it follows from Lemma 1.1 that $\ker f = \ker \varphi$. Therefore, the map $\bar{\varphi}$ is well-defined and injective. For $\bar{v}, \bar{w} \in \bar{V}$, we have $(\bar{\varphi}(\bar{v}), \bar{\varphi}(\bar{w})) = (\varphi(v), \varphi(w)) = rf(v,w) = r\bar{f}(\bar{v}, \bar{w})$, so $\bar{\varphi}$ is a similarity transformation of ratio $\sqrt{r}$. □

Let V and V' be inner product spaces and let $\Phi \subseteq V$ and $\Phi' \subseteq V'$. We write

$$(V, \Phi) \sim (V', \Phi')$$

to mean that there exists a bijective similarity transformation $\varphi : V \to V'$ such that $\varphi(\Phi) = \Phi'$. The relation $\sim$ is an equivalence relation on the class of all pairs (V, Φ), where V is an inner product space and $\Phi \subseteq V$. If $(V, \Phi) \sim (V', \Phi')$ we say that (V, Φ) is *similarly equivalent* to (V', Φ').

Let $\{V_i\}_{i\in I}$ be a family of inner product spaces and let Φ_i be a subset of V_i for each $i \in I$. Given an inner product space V with subset Φ, we write

$$(V, \Phi) \sim \bigoplus_{i\in I}(V_i, \Phi_i)$$

to mean that there exist $\Phi_i' \subseteq V_i' \leq V$ $(i \in I)$ such that $\Phi = \bigcup_i \Phi_i'$, $V = \dot{\sum}_i V_i'$ (internal orthogonal direct sum), and $(V_i', \Phi_i') \sim (V_i, \Phi_i)$ for each $i \in I$.

3. Symmetrized tensors

Fix positive integers n and m and set $\Gamma_{n,m} = \{\gamma \in \mathbf{Z}^n \mid 1 \leq \gamma_i \leq m\}$. Fix a subgroup G of the symmetric group S_n. A right action of G on the set $\Gamma_{n,m}$ is given by $\gamma\sigma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$ $(\gamma \in \Gamma_{n,m}, \sigma \in G)$. The *stabilizer* of $\gamma \in \Gamma_{n,m}$ is the set $G_\gamma = \{\sigma \in G \mid \gamma\sigma = \gamma\}$.

Let V be an inner product space of dimension m and let $\{e_i \mid 1 \leq i \leq m\}$ be an orthonormal basis for V. The inner product on V induces an inner product on $V^{\otimes n}$ (the nth tensor power of V) and, with respect to this inner product, the set $\{e_\gamma \mid \gamma \in \Gamma_{n,m}\}$ is an orthonormal basis for $V^{\otimes n}$, where $e_\gamma = e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_n}$.

The space $V^{\otimes n}$ is a (left) $\mathbf{C}G$-module with action given by $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ $(\sigma \in G, \gamma \in \Gamma_{n,m})$, extended linearly. The inner product on $V^{\otimes n}$ is G-invariant, which is to say $(\sigma v, \sigma w) = (v, w)$ for all $\sigma \in G$ and all $v, w \in V^{\otimes n}$.

Let $\chi \in \mathrm{Irr}(G)$ (the set of irreducible characters of G). The *symmetrizer* corresponding to χ is

$$s^\chi = \frac{\chi(e)}{|G|} \sum_{\sigma\in G} \chi(\sigma^{-1})\sigma \in \mathbf{C}G,$$

where e denotes the identity element of G. This element s^χ is the central idempotent of $\mathbf{C}G$ corresponding to χ [CR62, 33.8].

Let $\gamma \in \Gamma_{n,m}$. The *standard* (*decomposable*) *symmetrized tensor* corresponding to χ and γ is $e_\gamma^\chi = s^\chi e_\gamma$. The *orbital subspace* of $V^{\otimes n}$ corresponding to χ and γ, denoted V_γ^χ, is the span of the set $\Sigma = \Sigma_\gamma^\chi = \{e_{\gamma\sigma}^\chi \mid \sigma \in G\}$. The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces.

Next, we recall the definition of the coset space $\mathcal{C}_H^\chi$ corresponding to χ and a subgroup H of G [Hol04]. (This construction does not require G to be a subgroup of a symmetric group.)

Let H be a subgroup of G. The natural action of G on the set G/H of left cosets of H induces a $\mathbf{C}G$-module structure on the vector space $\mathbf{C}(G/H)$ with basis G/H.

Let $\chi \in \mathrm{Irr}(G)$. A well-defined G-invariant positive semidefinite Hermitian form B_H^χ on $\mathbf{C}(G/H)$ is obtained by putting

$$B_H^\chi(aH, bH) = \frac{\chi(e)}{|H|} \sum_{h\in H} \chi(b^{-1}ah)$$

$(a, b \in G)$ and extending linearly to $\mathbf{C}(G/H)$. The *coset space* corresponding to χ and H is the space $\mathcal{C}_H^\chi = \mathbf{C}(G/H)/\ker B_H^\chi$. By Lemma 1.1, B_H^χ induces an inner product $\bar{B}_H^\chi$ on $\mathcal{C}_H^\chi$. We have

$$\dim \mathcal{C}_H^\chi = \chi(e)(\chi, 1)_H = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h),$$

where, as usual, $(\varphi, \psi)_H = |H|^{-1} \sum_{h\in H} \varphi(h)\psi(h^{-1})$ for functions $\varphi, \psi : G \to \mathbf{C}$.

We refer to $\overline{aH} \in \mathcal{C}_H^\chi$ $(a \in G)$ as a *standard vector*. Put $\Psi = \Psi_H^\chi = \{\overline{aH} \mid a \in G\} \subseteq \mathcal{C}_H^\chi$.

3.1 Theorem. *For $\gamma \in \Gamma_{n,m}$, we have $(V_\gamma^\chi, \Sigma) \sim (\mathcal{C}_{G_\gamma}^\chi, \Psi)$.*

Proof. Let $\gamma \in \Gamma_{n,m}$ and put $H = G_\gamma$. The map $G/H \to V_\gamma^\chi$, $\sigma H \mapsto e_{\gamma\sigma^{-1}}^\chi$, is well-defined and it induces a surjective linear map $\varphi : \mathbf{C}(G/H) \to V_\gamma^\chi$. For $\sigma, \tau \in G$, we have

$$(3.1.1) \qquad (\varphi(\sigma H), \varphi(\tau H)) = (e_{\gamma\sigma^{-1}}^\chi, e_{\gamma\tau^{-1}}^\chi) = \frac{\chi(e)}{|G|} \sum_{\mu \in H} \chi(\tau^{-1}\sigma\mu) = rB_H^\chi(\sigma H, \tau H),$$

where $r = |G : H|^{-1}$ and where the second equality is from [Fre73, p. 339] (with $\bar{\chi}$ in place of χ). Using linearity we get $(\varphi(x), \varphi(y)) = rB_H^\chi(x, y)$ for all $x, y \in \mathbf{C}(G/H)$, so by Proposition 2.3, the induced map $\bar{\varphi} : \mathcal{C}_H^\chi \to V_\gamma^\chi$ given by $\bar{\varphi}(\bar{x}) = \varphi(x)$ is a well-defined bijective similarity transformation. Moreover $\bar{\varphi}(\Psi) = \Sigma$, so the claim follows. □

According to Theorem 3.1, every orbital subspace can be identified with a coset space in such a way that the standard symmetrized tensors in the orbital subspace identify, in an angle preserving and relative length preserving manner, with the standard vectors in the coset space.

The following result says that, conversely, every coset space can be similarly identified with an orbital subspace. The statement requires some explanation: Let $G = \{g_1, \ldots, g_n\}$ be a finite group. The **Cayley embedding** of G in the symmetric group $S_{|G|}$ is the monomorphism $\varphi : G \to S_{|G|}$ given by $\varphi(g) = \lambda_g$, with $\lambda_g : G \to G$ defined by $\lambda_g(a) = ga$. Here, we regard λ_g as an element of $S_{|G|}$ by using the identification $\{1, \ldots, n\} \leftrightarrow G$, $i \leftrightarrow g_i$. Using this same identification, we write γ_{g_i} to mean γ_i for $\gamma \in \Gamma_{|G|,m}$. Hence, $\gamma g = (\gamma_{gg_1}, \ldots, \gamma_{gg_n})$ for $\gamma = (\gamma_{g_1}, \ldots, \gamma_{g_n}) = (\gamma_1, \ldots, \gamma_n) \in \Gamma_{|G|,m}$ and $g \in G$.

3.2 Corollary. *Assume that $m = \dim V \geq 2$. Let G be a finite group, let $\chi \in \mathrm{Irr}(G)$, and let $H \leq G$. Identifying G as a subgroup of $S_{|G|}$ via the Cayley embedding, we have $(\mathcal{C}_H^\chi, \Psi) \sim (V_\gamma^\chi, \Sigma)$, where $\gamma \in \Gamma_{|G|,m}$ is defined by putting γ_g equal to 1 or 2 according as $g \in H$ or $g \notin H$.*

Proof. We have $H = G_\gamma$, so the claim follows from Theorem 3.1. □

Let G be a subgroup of S_n. The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces, so in order to study the geometry of the full set of standard symmetrized tensors associated with G, it is enough to study the set of standard symmetrized tensors in each orbital subspace of $V^{\otimes n}$. For this, it is enough, due to Theorem 3.1, to study each pair $(\mathcal{C}_H^\chi, \Psi)$ with $\chi \in \mathrm{Irr}(G)$ and $H \leq G$, although studying just those pairs with $H = G_\gamma$ for some $\gamma \in \Gamma_{n,m}$ would suffice.

Now let G be an arbitrary finite group and suppose that we wish to study the standard symmetrized tensors associated with *every* embedding of G in a symmetric group. Since every subgroup of G is a stabilizer in the case of the Cayley embedding (see Corollary 3.2), we need to study *all* pairs $(\mathcal{C}_H^\chi, \Psi)$ with $\chi \in \mathrm{Irr}(G)$ and $H \leq G$. For this, a concrete realization of each $(\mathcal{C}_H^\chi, \Psi)$ would be useful, so we suggest the following problem (see the end of Section 2 for notation).

3.3 Problem. Let G be a finite group. For each $\chi \in \mathrm{Irr}(G)$ and each $H \leq G$ find positive integers $n_1, \dots, n_t$ and subsets $\Phi_i \subseteq \mathbf{C}^{n_i}$ $(1 \leq i \leq t)$ such that $(\mathcal{C}_H^\chi, \Psi) \sim \bigoplus_i (\mathbf{C}^{n_i}, \Phi_i)$.

By way of illustration, we provide in Section 5 a solution to this problem in the case where G is a dihedral group (see Theorem 5.1).

3.4 *Remark.* Suppose a solution to Problem 3.3 for fixed χ and H is given. It is then a routine exercise to realize the set Ψ of standard vectors in $\mathcal{C}_H^\chi$ as a set of vectors in a *single* space $\mathbf{C}^n$:

Since the elements of Ψ all have the same length, it follows that, for each i, the elements of Φ_i all have the same length as well. So by scaling, if necessary, we may arrange for each Φ_i to consist of unit vectors. Then, since again the elements of Ψ all have the same length, the assumed bijective similarity transformations in the definition of the direct sum are all forced to have the same ratio and they can therefore form the component functions of a single bijective similarity transformation to show that $(\mathcal{C}_H^\chi, \Psi) \sim (\bigoplus_i \mathbf{C}^{n_i}, \bigcup_i \iota_i(\Phi_i)) \sim (\mathbf{C}^n, \Phi)$. Here, $\iota_i : \mathbf{C}^{n_i} \to \bigoplus_j \mathbf{C}^{n_j}$ is the ith injection, $n = \sum_i n_i$, and Φ is the image of $\bigcup_i \iota_i(\Phi_i)$ under the natural isomorphism $\bigoplus_i \mathbf{C}^{n_i} \to \mathbf{C}^n$.

While realizing the set Ψ in a single space $\mathbf{C}^n$ has a certain appeal, the carrying out of the procedure just described adds complexity without providing additional information about the geometry, so this is why we have allowed for more flexibility in the statement of the problem.

4. General results

Let G be a finite group. In this section, we obtain some general results about the pairs $(\mathcal{C}_H^\chi, \Psi)$ with $\chi \in \mathrm{Irr}(G)$ and $H \leq G$ (see Problem 3.3 and the remarks preceding it).

For $a, g \in G$ and $H \leq G$, we use the notation $a^g = g^{-1}ag$ and $H^g = \{h^g \,|\, h \in H\}$. The following theorem shows that Problem 3.3 can be considered solved for arbitrary $H \leq G$ once it has been solved for all H in a set of representatives for the conjugacy classes of subgroups of G.

4.1 Theorem. *Let $\chi \in \mathrm{Irr}(G)$ and let $H \leq G$. For every $g \in G$, we have $(\mathcal{C}_H^\chi, \Psi_H^\chi) \sim (\mathcal{C}_{H^g}^\chi, \Psi_{H^g}^\chi)$.*

Proof. Let $g \in G$. There is a well-defined linear map $\varphi : \mathbf{C}(G/H) \to \mathcal{C}_{H^g}^\chi$ uniquely determined by $\varphi(aH) = \overline{a^g H^g}$ $(a \in G)$. For $a, b \in G$, we have

$$\bar{B}_{H^g}^\chi(\varphi(aH), \varphi(bH)) = \bar{B}_{H^g}^\chi(\overline{a^g H^g}, \overline{b^g H^g}) = \frac{\chi(e)}{|H^g|} \sum_{h \in H} \chi((b^g)^{-1} a^g h^g)$$
$$= \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(b^{-1}ah) = B_H^\chi(aH, bH),$$

where the third equality follows from the fact that $G \to G$ by $x \mapsto x^g$ is an automorphism and then the fact that χ is constant on conjugacy classes. Using linearity, we get $\bar{B}_{H^g}^\chi(\varphi(x), \varphi(y)) = B_H^\chi(x, y)$ for all $x, y \in \mathbf{C}(G/H)$. Now φ is surjective, so by Proposition 2.3 the induced map $\bar{\varphi} : \mathcal{C}_H^\chi \to \mathcal{C}_{H^g}^\chi$ is a well-defined bijective similarity transformation. Moreover, $\bar{\varphi}(\Psi_H^\chi) = \Psi_{H^g}^\chi$, so the claim follows. □

For a $\mathbf{C}G$-module M and $x \in M$, put $G_x = \{g \in G \,|\, gx = x\}$ (stabilizer of x) and $Gx = \{gx \,|\, g \in G\}$ (orbit of x).

4.2 Theorem. *Let M be a simple $\mathbf{C}G$-module with a G-invariant inner product, let $\chi \in \mathrm{Irr}(G)$ be the character of G afforded by M, and let $0 \neq m_1 \in M$. If H is a subgroup of G with $H \subseteq G_{m_1}$ and $(\chi, 1)_H = 1$, then $(\mathcal{C}_H^\chi, \Psi) \sim (M, Gm_1)$.*

Proof. Let H be a subgroup of G with $H \subseteq G_{m_1}$ and $(\chi, 1)_H = 1$. We have $M_H = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_t$, where the M_i are simple $\mathbf{C}H$-submodules of M_H with $M_1 = \mathbf{C}m_1$ and $M_i \not\cong M_1$ for all $i \neq 1$. Let $e_H = |H|^{-1} \sum_{h \in H} h$. Then e_H is the central idempotent of $\mathbf{C}H$ corresponding to the trivial $\mathbf{C}H$-module, so that $e_H M_1 = M_1$ and $e_H M_i = 0$ for $i \neq 1$.

Put $N = \sum_{i \neq 1} M_i$. Using the G-invariance of the inner product on M, we get

$$(N, M_1) = (N, e_H M_1) = \frac{1}{|H|} \sum_{h \in H} (N, hM_1)$$
$$= \frac{1}{|H|} \sum_{h \in H} (h^{-1}N, M_1) = (e_H N, M_1) = (0, M_1) = 0.$$

Since $H \subseteq G_{m_1}$, we get a well-defined linear map $\varphi : \mathbf{C}(G/H) \to M$ uniquely determined by $\varphi(aH) = am_1$ $(a \in G)$, which is seen to be a $\mathbf{C}G$-homomorphism.

We claim that $(\varphi(x), \varphi(y)) = rB_H^\chi(x, y)$ for all $x, y \in \mathbf{C}(G/H)$, where $r = (m_1, m_1)/\chi(e)$. Let $x, y \in \mathbf{C}(G/H)$. Due to the linearity and G-invariance of the forms and the fact that φ is a $\mathbf{C}G$-homomorphism, we may assume that $x = aH$ and $y = H$ for some $a \in G$.

Extend the basis $\{m_1\}$ of M_1 to a basis $B = \{m_1, m_2, \dots, m_n\}$ of M by choosing a basis for each M_i and forming their union. Let $\alpha : G \to \mathrm{GL}_n(\mathbf{C})$, $g \mapsto [\alpha_{ij}(g)]$, be the matrix representation of G afforded by M relative to B.

On the one hand,

$$(\varphi(x), \varphi(y)) = (am_1, m_1) = \sum_i \alpha_{i1}(a)(m_i, m_1) = \alpha_{11}(a)(m_1, m_1),$$

since $(M_1, N) = 0$. On the other hand,

$$\begin{aligned}\frac{|H|}{\chi(e)} B_H^\chi(x, y) &= \sum_{h\in H} \chi(ah) = \sum_{h\in H} \sum_i \alpha_{ii}(ah) = \sum_{h\in H} \sum_{i,j} \alpha_{ij}(a)\alpha_{ji}(h) \\ &= \sum_{i,j} \alpha_{ij}(a) \sum_{h\in H} \alpha_{ji}(h) = |H|\alpha_{11}(a),\end{aligned}$$

since $\sum_{h\in H} \alpha_{ji}(h) = \sum_{h\in H} \alpha_{11}(h^{-1})\alpha_{ji}(h) = |H|\delta_{1j}\delta_{1i}$ (Kronecker delta) by [Ser77, p. 14, Corollaries 2 and 3]. Therefore, $(\varphi(x), \varphi(y)) = rB_H^\chi(x, y)$, as claimed.

Now φ is nonzero (since $m_1 \neq 0$), so it is surjective (since M is simple). It then follows from Proposition 2.3 that the map $\bar\varphi : \mathcal{C}_H^\chi \to M$ given by $\bar\varphi(\bar x) = \varphi(x)$ is a well-defined bijective similarity transformation. Finally, we have $\bar\varphi(\overline{aH}) = am_1$ for each $a \in G$, so $\bar\varphi(\Psi) = Gm_1$ and the proof is complete. □

For an explanation of the notation in the following theorem, see the end of Section 2.

4.3 Theorem. *Let $\chi \in \mathrm{Irr}(G)$ and let A and K be subgroups of G such that $G = AK$ and $\chi(g) = 0$ for all $g \in G\backslash A$. We have*

$$(\mathcal{C}_H^\chi, \Psi_H^\chi) \sim \bigoplus_{i=1}^n (\mathcal{C}_K^\chi, \Psi_K^\chi),$$

where $H = A \cap K$ and $n = |G : A|$.

Proof. Since $G = (AK)^{-1} = K^{-1}A^{-1} = KA$, there exists a complete set $\{k_1, k_2, \dots, k_n\}$ of left coset representatives of A in G with $k_i \in K$ for each i. Fix $1 \leq i \leq n$ and put $C_i = \{k_i aH \mid a \in A\}$. The map $C_i \to \mathcal{C}_K^\chi$ given by $k_i aH \mapsto \overline{aK}$ is well defined and it induces a surjective linear map

$\varphi : \mathbf{C}C_i \to \mathcal{C}_K^\chi$. For $a_1, a_2 \in A$ we have

$$\begin{aligned}\frac{|H|}{\chi(e)} B_H^\chi(k_i a_1 H, k_i a_2 H) &= \sum_{h\in H} \chi((k_i a_2)^{-1}(k_i a_1)h) = \sum_{k \in K} \chi(a_2^{-1} a_1 k)\\ &= \frac{|K|}{\chi(e)} \bar{B}_K^\chi(\overline{a_1 K}, \overline{a_2 K})\\ &= \frac{|K|}{\chi(e)} \bar{B}_K^\chi(\varphi(k_i a_1 H), \varphi(k_i a_2 H)),\end{aligned}$$

so, by linearity of the forms, we get $\bar{B}_K^\chi(\varphi(x), \varphi(y)) = rB_H^\chi(x, y)$ for all $x, y \in \mathbf{C}C_i$, where $r = |K : H|^{-1}$. It follows from Proposition 2.3 that φ induces a well-defined bijective similarity transformation $\bar{\varphi} : \mathbf{C}C_i / \ker B' \to \mathcal{C}_K^\chi$ satisfying $\bar{\varphi}(x + \ker B') = \varphi(x)$ for all $x \in \mathbf{C}C_i$, where B' denotes the restriction of the form B_H^χ to $\mathbf{C}C_i$. Using Lemma 1.1(i) we get $\ker B' = \mathbf{C}C_i \cap \ker B_H^\chi$, so by an isomorphism theorem, $\mathbf{C}C_i / \ker B' \cong (\mathbf{C}C_i + \ker B_H^\chi)/\ker B_H^\chi = \overline{\mathbf{C}C_i} \subseteq \mathcal{C}_H^\chi$, with the isomorphism sending $x + \ker B'$ to $x + \ker B_H^\chi = \bar{x}$. Identifying $\mathbf{C}C_i/\ker B'$ with $\overline{\mathbf{C}C_i}$ in this way we have $\bar{\varphi} : \overline{\mathbf{C}C_i} \to \mathcal{C}_K^\chi$ by $\bar{\varphi}(\bar{x}) = \varphi(x)$. Since $\bar{\varphi}(\overline{C_i}) = \Psi_K^\chi$, we conclude that $(\overline{\mathbf{C}C_i}, \overline{C_i}) \sim (\mathcal{C}_K^\chi, \Psi_K^\chi)$.

Let $1 \le i, j \le n$ with $i \ne j$ and let $a_1, a_2 \in A$. For each $h \in H$, we have $(k_j a_2)^{-1}(k_i a_1)h = a_2^{-1} k_j^{-1} k_i a_1 h \notin A$ (since $k_j^{-1} k_i \notin A$), so

$$\bar{B}_H^\chi(\overline{k_i a_1 H}, \overline{k_j a_2 H}) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi((k_j a_2)^{-1}(k_i a_1)h) = 0,$$

since χ vanishes off of A. Therefore, $\overline{\mathbf{C}C_i}$ is orthogonal to $\overline{\mathbf{C}C_j}$ for $i \ne j$. Finally, $\Psi_H^\chi = \overline{G/H} = \bigcup_i \overline{C_i}$ and $\mathcal{C}_H^\chi = \overline{\mathbf{C}(G/H)} = \sum_i \overline{\mathbf{C}C_i}$. It follows that $\mathcal{C}_H^\chi$ is the internal orthogonal direct sum of the spaces $\overline{\mathbf{C}C_i}$, $1 \le i \le n$, and the proof is complete. □

5. Dihedral group

In this section, we solve Problem 3.3 in the case where G is a dihedral group and use the results to give a more conceptual proof of an earlier result (see Corollary 5.3).

Let n be an integer with $n \ge 3$. The *dihedral group of degree* n is the group $G = D_{2n}$ with presentation

$$G = \langle r, s \,|\, r^n = e, s^2 = e, srs = r^{-1}\rangle.$$

We have $G = \{r^k, sr^k \,|\, 0 \le k < n\}$ and the indicated elements are distinct. In particular, G has order $2n$.

The irreducible characters of G are given in the following table [Ser77, pp. 37–38]:

	r^k	sr^k	
ψ_0	1	1	
ψ_1	1	-1	
ψ_2	$(-1)^k$	$(-1)^k$	(n even)
ψ_3	$(-1)^k$	$(-1)^{k+1}$	(n even)
χ_j	$2\cos(2\pi kj/n)$	0	$(1 \leq j < n/2)$.

Let $1 \leq j < n/2$. The character χ_j is afforded by the representation $\rho_j : G \to \mathrm{GL}_2(\mathbf{C})$ uniquely determined by

$$\rho_j(r) = \begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix}, \qquad \rho_j(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $E_j = \mathbf{C}^2$ be the $\mathbf{C}G$-module affording ρ_j.

Let m be a positive integer. The set

$$B_m = \{(\cos(2\pi k/m), \sin(2\pi k/m)) \,|\, k \in \mathbf{Z}\} \subseteq \mathbf{R}^2$$

is the set of vertices of a regular m-gon. Below, we view B_m as a subset of $\mathbf{C}$ (by identifying $\mathbf{R}^2$ with $\mathbf{C}$), but also as a subset of $\mathbf{C}^2$, relying on the context to make the meaning clear.

Denote by ν either 4 or 2 according as n is even or odd (so ν is the number of linear characters ψ_j of G). The kernel of a character χ of G is defined by $\ker\chi = \{g \in G \,|\, \chi(g) = \chi(e)\}$; it equals the kernel of the representation of G affording χ.

5.1 Theorem. *Let H be a subgroup of $G = D_{2n}$.*

(i) *Let $0 \leq j < \nu$ and put $\chi = \psi_j$. If $H \nsubseteq \ker\chi$, then $\mathcal{C}_H^\chi = \{0\}$. If $H \subseteq \ker\chi$, then*

$$(\mathcal{C}_H^\chi, \Psi) \sim \begin{cases} (\mathbf{C}, B_1), & j = 0, \\ (\mathbf{C}, B_2), & j \neq 0. \end{cases}$$

(ii) *Let $1 \leq j < n/2$ and put $\chi = \chi_j$. We have $\ker\chi = \langle r^{n'} \rangle$, where $n' = n/\gcd(n, j)$. Put $J = H \cap C_n$, where $C_n = \langle r \rangle$. If $J \nsubseteq \ker\chi$, then $\mathcal{C}_H^\chi = \{0\}$. If $J \subseteq \ker\chi$, then*

$$(\mathcal{C}_H^\chi, \Psi) \sim \begin{cases} (\mathbf{C}^2, B_{n'}), & H \nsubseteq C_n, \\ (\mathbf{C}^2, B_{n'}) \oplus (\mathbf{C}^2, B_{n'}), & H \subseteq C_n. \end{cases}$$

Proof. (i) If $H \nsubseteq \ker\chi$, then $\dim\mathcal{C}_H^\chi = \chi(e)(\chi, 1)_H = 0$, so $\mathcal{C}_H^\chi = \{0\}$. Assume that $H \subseteq \ker\chi$. View $\mathbf{C}$ as the $\mathbf{C}G$-module affording χ and note that the inner product on $\mathbf{C}$ is G-invariant (cf. proof of (ii)). We have $(\chi, 1)_H = 1$ and $H \subseteq G_{m_1}$, where $m_1 = 1 \in \mathbf{C}$. Moreover, $Gm_1 = \{1\} = B_1$ if $j = 0$ and $Gm_1 = \{\pm 1\} = B_2$ if $j \neq 0$, so the claim follows from Theorem 4.2.

(ii) First, $\ker\chi \subseteq C_n$ and, for any integer k,

$$r^k \in \ker\chi \iff 2\cos(2\pi kj/n) = 2 \iff kj'/n' = kj/n \in \mathbf{Z} \iff n'|k,$$

where $j' = j/\gcd(n,j)$. Therefore, $\ker\chi = \langle r^{n'}\rangle$.

Assume that $J \nsubseteq \ker\chi$. We have $\dim \mathcal{C}_H^\chi = \chi(e)(\chi,1)_H \leq \chi(e)(\chi,1)_J = 0$, the last equality from [Isa94, 6.7]. Therefore, $\mathcal{C}_H^\chi = \{0\}$.

Now assume that $J \subseteq \ker\chi$. We have

$$(5.1.1) \qquad (\chi,1)_H = \frac{1}{|H|}\sum_{h\in H}\chi(h) = \frac{1}{|H|}\sum_{a\in J}\chi(a) = 2/|H:J|,$$

since χ vanishes off of C_n and $\chi(a) = 2$ for all $a \in J$.

Put $\rho = \rho_j$. For $g \in G$, the matrix $\rho(g)$ is orthogonal, so for $x, y \in E_j = \mathbf{C}^2$, we have (using $*$ for conjugate transpose)

$$(gx, gy) = (\rho(g)y)^*\rho(g)x = y^*\rho(g)^*\rho(g)x = y^*x = (x,y).$$

Therefore, the inner product on E_j is G-invariant.

Assume that $H \nsubseteq C_n$. Then $H = J \rtimes T$, where $T = \langle t\rangle$ for some $t \in G\backslash C_n$. Now T has order two, so $(\chi,1)_H = 1$ by Equation 5.1.1. The element t acts on $\mathbf{R}^2 \subseteq E_j$ as a reflection and therefore it fixes some nonzero $m_1 \in E_j$. Since $J \subseteq \ker\chi$, every element of J fixes m_1 as well, implying that $H \subseteq G_{m_1}$. Since r acts on $\mathbf{R}^2$ by counterclockwise rotation through the angle $2\pi j/n = 2\pi j'/n'$, it follows that $Gm_1 = C_nTm_1 = C_nm_1$ is the set of vertices of a regular n'-gon. By Theorem 4.2, $(\mathcal{C}_H^\chi, \Psi) \sim (E_j, Gm_1) \sim (\mathbf{C}^2, B_{n'})$, where the last similarity is obtained by applying a rotation and/or a homothety to send Gm_1 to $B_{n'}$.

Now assume that $H \subseteq C_n$. Put $K = HT \leq G$, where $T = \langle s\rangle$. By the preceding paragraph, $(\mathcal{C}_K^\chi, \Psi_K^\chi) \sim (\mathbf{C}^2, B_{n'})$. Therefore, Theorem 4.3 with $A = C_n$ establishes the claim. □

We end this section by giving an application of Theorem 5.1, which requires the following (geometrically evident) fact.

5.2 Lemma. *The set B_m contains a pair of orthogonal vectors if and only if m is divisible by 4.*

Proof. By symmetry B_m contains a pair of orthogonal vectors precisely when it contains a vector orthogonal to $(1,0)$, that is, if and only if it contains $(0,1)$. But the equation $(0,1) = (\cos(2\pi k/m), \sin(2\pi k/m))$ holds for some integer k precisely when m is divisible by 4, so the claim follows. □

A finite group G is an *o-basis group* [Hol04] if for each $\chi \in \mathrm{Irr}(G)$ and each $H \leq G$ there exists an orthogonal basis of $\mathcal{C}_H^\chi$ consisting entirely of standard vectors (such a basis is called an *o-basis*). If G is an o-basis group, then it follows from Theorem 3.1 that for every embedding $G \hookrightarrow S_n$ of G in a symmetric group, the space $V^{\otimes n}$ has an orthogonal basis consisting entirely of standard symmetrized tensors (relative to the image of G under the embedding).

The following result appears in [Hol04, Corollaries 2.2 and 3.2] (see also [HT92, Corollary 3.3]).

5.3 Corollary. *The dihedral group D_{2n} is an o-basis group if and only if $n = 2^k$ for some positive integer k.*

Proof. Assume that $G = D_{2n}$ is an o-basis group. Let m be a positive integer with $m > 2$ and assume that m divides n. Then $1 \leq j < n/2$, where $j = n/m$. Letting $H = \langle s \rangle$ and $\chi = \chi_j$ in Theorem 5.1 we get $(\mathcal{C}_H^\chi, \Psi) \sim (\mathbf{C}^2, B_m)$. Since Ψ contains a pair of orthogonal vectors, so does B_m, and therefore m is divisible by 4 by Lemma 5.2. It follows that $n = 2^k$ for some positive integer k.

Now assume that $n = 2^k$ for some positive integer k and note that $n \geq 4$ (since we took $n \geq 3$ in the definition of D_{2n}). Let $\chi \in \mathrm{Irr}(G)$ and let $H \leq G$. If $\chi = \psi_j$ for some $0 \leq j < \nu$, then $\dim \mathcal{C}_H^\chi \leq 1$ by Theorem 5.1, so $\mathcal{C}_H^\chi$ has an o-basis. Therefore, we may assume that $\chi = \chi_j$ for some $1 \leq j < n/2$. Put $J = H \cap C_n$. If $J \nsubseteq \ker \chi$, then $\mathcal{C}_H^\chi$ is $\{0\}$ by Theorem 5.1 and it therefore has an o-basis. Assume, on the other hand, that $J \subseteq \ker \chi$. Now n' is divisible by 4, so $B_{n'}$ contains a pair of orthogonal vectors by Lemma 5.2. It follows from Theorem 5.1 that $\mathcal{C}_H^\chi$ has an o-basis. Therefore, G is an o-basis group. □

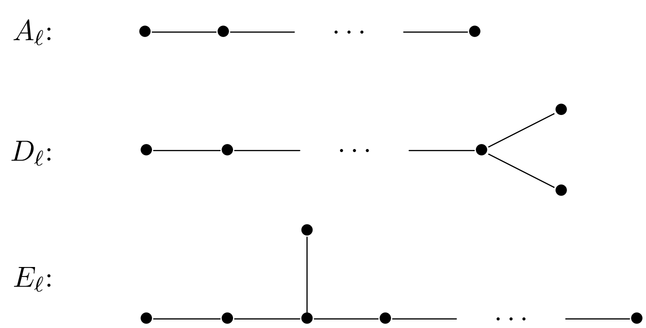
6. Root system

In [TS12], Torres and Silva construct, for each integer $\ell \geq 3$, an orbital subspace having the property that the standard symmetrized tensors in the subspace form an irreducible root system of type A_ℓ. It is natural to ask which of the irreducible root systems can be realized as the set of standard symmetrized tensors in some orbital subspace. We show in this section that it is precisely the simply laced irreducible root systems that can be so realized.

Let E be an inner product space, let Φ be a subset of E, and denote by $E_{\mathbf{R}}$ the $\mathbf{R}$-span of Φ. We refer to the pair (E, Φ) as an *irreducible root system* if $\Phi \subseteq E_{\mathbf{R}}$ is an irreducible root system in the sense of [Hum72, 9.2, 10.4] and the map $\mathbf{C} \otimes_{\mathbf{R}} E_{\mathbf{R}} \to E$ induced by the inclusion map $E_{\mathbf{R}} \hookrightarrow E$ is an isomorphism.

Let E' be another inner product space and let Φ' be a subset of E'. We write $(E, \Phi) \cong (E', \Phi')$ to mean that there exists a vector space isomorphism $\varphi : E \to E'$ such that $\varphi(\Phi) = \Phi'$ and $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$, where $\langle \alpha, \beta \rangle := 2(\alpha, \beta)/(\beta, \beta)$. If (E, Φ) is an irreducible root system and $(E, \Phi) \cong (E', \Phi')$, then (E', Φ') is an irreducible root system as well and the irreducible root systems $\Phi \subseteq E_{\mathbf{R}}$ and $\Phi' \subseteq E'_{\mathbf{R}}$ are *isomorphic* in the sense of [Hum72, 9.2]. It is immediate that $(E, \Phi) \sim (E', \Phi')$ implies $(E, \Phi) \cong (E', \Phi')$.

An irreducible root system (E, Φ) is *simply laced* if the elements of Φ all have the same length, that is, $\|\alpha\| = \|\beta\|$ for all $\alpha, \beta \in \Phi$. The simply laced irreducible root systems are the ones of types A_ℓ ($\ell \geq 1$), D_ℓ ($\ell \geq 4$), E_ℓ ($\ell = 6, 7, 8$). Their corresponding Dynkin diagrams are shown below (each with ℓ vertices):



6.1 Theorem. *Let (E, Φ) be an irreducible root system and assume that $m = \dim V \geq 2$. The following are equivalent:*

(i) *There exists $n \in \mathbf{N}$ and a subgroup G of S_n such that $(E, \Phi) \cong (V_\gamma^\chi, \Sigma)$ for some $\chi \in \mathrm{Irr}(G)$ and some $\gamma \in \Gamma_{n,m}$.*
(ii) *(E, Φ) is simply laced.*

Proof. Assume that (i) holds so that there exists $\varphi : E \to V_\gamma^\chi$ as in the definition. For $\alpha, \beta \in \Phi$, we have $\langle \alpha, \beta \rangle \in \mathbf{Z}$ by a root system axiom, so $(\alpha, \beta), (\varphi(\alpha), \varphi(\beta)) \in \mathbf{R}$, implying

$$\frac{(\varphi(\alpha), \varphi(\alpha))}{(\varphi(\beta), \varphi(\beta))} = \frac{\langle \varphi(\alpha), \varphi(\beta) \rangle}{\langle \varphi(\beta), \varphi(\alpha) \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \alpha \rangle} = \frac{(\alpha, \alpha)}{(\beta, \beta)}.$$

Since the elements of Σ all have the same length (Equation 3.1.1), the same is true for Φ and (ii) holds.

Now assume that (ii) holds. We claim that there exists a finite group G such that $(E, \Phi) \sim (\mathcal{C}_H^\chi, \Psi)$ for some $\chi \in \mathrm{Irr}(G)$ and some $H \leq G$. Once this is established, (i) will follow from Corollary 3.2 and the proof will be complete.

If (E, Φ) is of type A_1, then $\Phi = \{\alpha, -\alpha\}$ for some $\alpha \in E$, so $(E, \Phi) \sim (\mathbf{C}, B_2)$ and our claim follows from Theorem 5.1(i) by letting $G = D_6$, $\chi = \psi_1$, and $H = \{e\}$. Similarly, if (E, Φ) is of type A_2, then Φ is the set of vertices of a regular hexagon, so $(E, \Phi) \sim (\mathbf{C}^2, B_6)$ and our claim follows from Theorem 5.1(ii) by letting $G = D_{12}$, $\chi = \chi_1$, and $H = \langle s \rangle$. Therefore, we may assume that (E, Φ) is neither of type A_1 nor of type A_2.

Let G be the Weyl group of $\Phi \subseteq E_{\mathbf{R}}$. According to the proof of [Hum72, p. 53, Lemma B], $E_{\mathbf{R}}$ is an absolutely simple $\mathbf{R}G$-module. Therefore, E is a simple $\mathbf{C}G$-module. The elements of G are reflections of $E_{\mathbf{R}}$, so the inner product on $E_{\mathbf{R}}$ is G-invariant, as is the induced inner product on E.

The map $\varepsilon : G \to \mathbf{C}^*$ given by $\varepsilon(\sigma) = (-1)^{l(\sigma)}$ is a homomorphism, where $l(\sigma)$ is the length of σ [Hum72, p. 54, Exercise 6]. Denote by $\mathbf{C}_\varepsilon$ the corresponding $\mathbf{C}G$-module. The tensor product $E_\varepsilon = \mathbf{C}_\varepsilon \otimes_{\mathbf{C}} E$ is a simple

$\mathbf{C}G$-module. After identifying the vector space E_ε with E as usual, we can describe the action of G on E_ε as being given by the formula $\sigma \cdot x = \varepsilon(\sigma)\sigma x$ ($\sigma \in G$, $x \in E_\varepsilon$). The irreducible character of G afforded by E_ε is $\chi = \varepsilon\psi$, where ψ is the character of G afforded by E.

Let $\triangle$ be a base for $\Phi \subseteq E_{\mathbf{R}}$. In the Dynkin diagram of $\triangle$ there exists a vertex $\alpha \in \triangle$ that is adjacent to precisely one other vertex $\beta \in \triangle$ (since (E, Φ) is not of type A_1). Then $\sigma_\alpha\alpha = -\alpha$, $\sigma_\alpha\beta = \alpha + \beta$, and $\sigma_\alpha\gamma = \gamma$ for $\gamma \in \triangle\backslash\{\alpha, \beta\}$, where σ_α is the reflection in the hyperplane orthogonal to α. Therefore, $\chi(\sigma_\alpha) = 2 - \ell$, where $\ell = |\triangle|$.

Put $H = \langle\sigma_\alpha\rangle \leq G$. We have $\sigma_\alpha \cdot \alpha = \varepsilon(\sigma_\alpha)\sigma_\alpha\alpha = \alpha$, so $H \subseteq G_\alpha$. Also, $(\chi, 1)_H = \frac{1}{|H|}(\chi(1) + \chi(\sigma_\alpha)) = \frac{1}{2}(\ell + (2 - \ell)) = 1$.

In general, the Weyl group G acts transitively on the set of those roots having a fixed length [Hum72, p. 53, Lemma C]. Therefore, since (E, Φ) is simply laced, we have $G\alpha = \Phi$, implying $G \cdot \alpha \cup -G \cdot \alpha = \Phi$. We are assuming that Φ is neither of type A_1 nor of type A_2, so there exists a vertex $\gamma \in \triangle$ of the Dynkin diagram of $\triangle$ that is not adjacent to α. Therefore, $\sigma = \sigma_\alpha\sigma_\gamma$ has length two and $-\alpha = \varepsilon(\sigma)\sigma\alpha = \sigma \cdot \alpha \in G \cdot \alpha$, so $\Phi = G \cdot \alpha$.

By Theorem 4.2 with $M = E_\varepsilon$ and $m_1 = \alpha$, we get $(E, \Phi) \sim (\mathcal{C}_H^\chi, \Psi)$. This establishes our claim and finishes the proof. □

6.2 ***Remark.*** Let the notation be as in Theorem 6.1. In the proof that (ii) implies (i), after reducing to the case where the irreducible root system is neither of type A_1 nor of type A_2, G is taken to be the image of the Weyl group of $\Phi \subseteq E_{\mathbf{R}}$ under the Cayley embedding (so $G \leq S_{|G|}$). In view of Theorem 3.1, it is sufficient though for G to be isomorphic to the Weyl group of $\Phi \subseteq E_{\mathbf{R}}$ and have the property that $H = G_\gamma$ for some $\gamma \in \Gamma_{n,m}$. In the case that (E, Φ) is of type A_ℓ, the Weyl group can be identified with $G = S_n$ (viewed as a subgroup of itself), where $n = \ell + 1$, and then the proof of Theorem 6.1 shows that we can let $H = \langle(1, 2)\rangle$, so that $H = G_\gamma$, where $\gamma = (1, 1, 2, 3, \ldots, \ell)$ (this requires $m = \dim V \geq \ell$). The resulting orbital subspace containing a copy of Φ is then the same as that constructed in [TS12, Theorem 14].

References

[BPR03] C. Bessenrodt, M. R. Pournaki, and A. Reifegerste, *A note on the orthogonal basis of a certain full symmetry class of tensors*, Linear Algebra Appl. **370** (2003), 369–374. MR 1994340 (2004f:15052)

[CR62] Charles W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962. MR 0144979 (26 #2519)

[DP99] M. R. Darafsheh and N. S. Poursalavati, *Orthogonal basis of the symmetry classes of tensors associated with the direct product of permutation groups*, Pure Math. Appl. **10** (1999), no. 3, 241–248. MR 1752864 (2001f:15030)

[Fre73] Ralph Freese, *Inequalities for generalized matrix functions based on arbitrary characters*, Linear Algebra and Appl. **7** (1973), 337–345. MR 0340273 (49 #5028)

[Hol95] Randall R. Holmes, *Orthogonal bases of symmetrized tensor spaces*, Linear and Multilinear Algebra **39** (1995), no. 3, 241–243. MR 1365442 (96k:15015)

[Hol04] ———, *Orthogonality of cosets relative to irreducible characters of finite groups*, Linear Multilinear Algebra **52** (2004), no. 2, 133–143. MR 2033134 (2004j:20011)

[HT92] Randall R. Holmes and Tin Yau Tam, *Symmetry classes of tensors associated with certain groups*, Linear and Multilinear Algebra **32** (1992), no. 1, 21–31. MR 1198818 (93j:20019)

[Hum72] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842 (48 #2197)

[Isa94] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications Inc., New York, 1994, Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. MR 1280461

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR 0450380 (56 #8675)

[TS12] Maria M. Torres and Pedro C. Silva, *Tensors, matchings and codes*, Linear Algebra Appl. **436** (2012), no. 6, 1606–1622. MR 2890942

[WG91] Bo Ying Wang and Ming Peng Gong, *A high symmetry class of tensors with an orthogonal basis of decomposable symmetrized tensors*, Linear and Multilinear Algebra **30** (1991), no. 1-2, 61–64. MR 1119469 (93f:15028)

Hank G. Harmon, Department of Mathematics and Statistics, Auburn University, Auburn AL, 36849, hgh0001@tigermail.auburn.edu

Randall R. Holmes, Department of Mathematics and Statistics, Auburn University, Auburn AL, 36849, holmerr@auburn.edu