# Homological Algebra II

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# 1 Global dimension

**1.1**. (Projective dimension)

Let $A$ be an $R$-module. The **projective dimension** of $A$, written $\mathrm{pd}(A)$, is the least nonnegative integer $d$ for which there exists a projective resolution of $A$ of the form

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to A \to 0,$$

unless there is no such resolution, in which case $\mathrm{pd}(A) = \infty$.

For instance, $A$ is projective if and only if $\mathrm{pd}(A) = 0$. And if $R = \mathbf{Z}$, then $\mathrm{pd}(\mathbf{Z}_2) = 1$ $(0 \to 2\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0)$.

Theorem. *For given $A$ and $d$, the following are equivalent:*

(i) $\mathrm{pd}(A) \le d$,

(ii) $\mathrm{Ext}_R^n(A, B) = 0$ *for all* $n > d$ *and all* $B$,

(iii) $\mathrm{Ext}_R^{d+1}(A, B) = 0$ *for all* $B$,

(iv) *if* $0 \to M_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ *is exact with each* $P_i$ *projective, then* $M_d$ *is projective.*

When $d = 0$, the equivalence (i) $\Leftrightarrow$ (iii) says that $A$ is projective if and only if $\mathrm{Ext}_R^1(A, B) = 0$ for all $B$, which is the theorem in I-12.3.

The proof of this theorem requires the following lemma, which is proved using the long exact Ext sequence and the technique of "dimension shifting."

Lemma. *If*

$$0 \to M_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

$(n \ge 1)$ *is exact with each* $P_i$ *projective, then* $\mathrm{Ext}_R^{i-n}(M_n, B) \cong \mathrm{Ext}_R^i(A, B)$ *for every* $R$*-module* $B$ *and every* $i \ge n + 1$.

Here is an application of the theorem. If $m$ is an integer and $R = \mathbf{Z}_{m^2}$, then the $R$-module $\mathbf{Z}_m$ has infinite projective dimension by (i)$\Rightarrow$(iv) since $\mathbf{Z}_m$ is not projective and there is a resolution

$$0 \to \mathbf{Z}_m \to \mathbf{Z}_{m^2} \to \mathbf{Z}_{m^2} \to \cdots \to \mathbf{Z}_{m^2} \to \mathbf{Z}_m \to 0$$

of arbitrary length with $\mathbf{Z}_m \to \mathbf{Z}_{m^2}$ the natural monomorphism and the other maps multiplication by $m$.

**1.2.** (Injective dimension)

Let $B$ be an $R$-module. The **injective dimension** of $B$, written $\mathrm{id}(B)$, is the least nonnegative integer $d$ for which there exists an injective resolution of $B$ of the form

$$0 \to B \to E^0 \to E^1 \to \cdots \to E^d \to 0,$$

unless there is no such resolution, in which case $\mathrm{id}(B) = \infty$.

For instance, if $B$ is injective, then $\mathrm{id}(B) = 0$, and if $R = \mathbf{Z}$, then $\mathrm{id}(\mathbf{Z}) = 1$ $(0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0)$.

THEOREM. *For given $B$ and $d$, the following are equivalent:*

(i) $\mathrm{id}(B) \leq d$,

(ii) $\mathrm{Ext}_R^n(A, B) = 0$ *for all* $n > d$ *and all* $A$,

(iii) $\mathrm{Ext}_R^{d+1}(A, B) = 0$ *for all* $A$,

(iv) *if* $0 \to B \to E^0 \to E^1 \to \cdots \to E^{d-1} \to M^d \to 0$ *is exact with each* $E^i$ *injective, then* $M^d$ *is injective.*

The proof of this theorem is almost identical to the proof of the theorem in the previous section.

**1.3.** (Global dimension)

The **left global dimension** of the ring $R$, written $\mathrm{lgd}(R)$, is the supremum of the set of projective dimensions of all (left) $R$-modules:

$$\mathrm{lgd}(R) = \sup\{\mathrm{pd}(A) \mid A \in R\text{-}\mathbf{mod}\}$$

For example, $\mathrm{lgd}(\mathbf{Z}) = 1$ since $\mathrm{pd}(\mathbf{Z}_2) = 1$ and for any $\mathbf{Z}$-module $A$, there exists an exact sequence $0 \to F_1 \to F_0 \to A \to 0$ with $F_i$ free.

THEOREM. *The following numbers are equal:*

(i) $\operatorname{lgd}(R)$,

(ii) $\sup\{\operatorname{id}(B) \mid B \in R\text{-}\boldsymbol{mod}\}$,

(iii) $\sup\{\operatorname{pd}(R/I) \mid I$ *is a left ideal of* $R\}$,

(iv) $\sup\{d \mid \operatorname{Ext}^d_R(A, B) \neq 0$ *for some* $A, B \in R\text{-}\boldsymbol{mod}\}$.

There is a corresponding notion of "right global dimension" of $R$, written $\operatorname{rgd}(R)$. It has been shown that for every $1 \leq m, n \leq \infty$ there exists a ring $R$ such that $\operatorname{lgd}(R) = m$ and $\operatorname{rgd}(R) = n$ (Jategaonkar, 1969). On the other hand, if $R$ is both left and right Noetherian, then $\operatorname{lgd}(R) = \operatorname{rgd}(R)$.

**1.4.** (Semisimple ring)

In this section, we show that the class of rings with zero global dimension (either left or right) is precisely the class of semisimple rings.

An $R$-module is **simple** if it has no nonzero proper submodules. An $R$-module is **semisimple** if it is a direct sum of simple submodules.

THEOREM. *An $R$-module $A$ is semisimple if and only if every exact sequence $0 \to B \to A \to C \to 0$ splits.*

A ring is **semisimple** if it is semisimple as a left module over itself (or equivalently, as a consequence of the next theorem and its identical right counterpart, as a right module over itself).

THEOREM (Wedderburn). *$R$ is semisimple if and only if $R \cong \prod_{i=1}^t \operatorname{Mat}_{n_i}(D_i)$, for some $t$, $n_i$, and division rings $D_i$.*

If $R$ is semisimple, then the $t$, $n_i$ and $D_i$ in the theorem are uniquely determined by $R$. Each ring $\operatorname{Mat}_{n_i}(D_i)$ is simple (i.e., has no nonzero proper two-sided ideals), so a semisimple ring is a finite direct product of simple rings. Conversely, if $R$ is a finite direct product of simple left (or right) Artinian rings, then $R$ is semisimple. (A ring is left Artinian if its left ideals satisfy the descending chain condition, or, equivalently, if each nonempty collection of its left ideals has a minimal element, that is, a left ideal that does not properly contain any other left ideal in the collection.)

Maschke's theorem says that if $G$ is a finite group and $K$ is a field of characteristic not a divisor of $|G|$, then the group ring $KG$ is semisimple.

THEOREM. *The following are equivalent:*

(i) *$R$ is semisimple,*

(ii) $\mathrm{lgd}(R) = 0$,

(iii) *every $R$-module is injective,*

(iv) *every $R$-module is projective,*

(v) *every short exact sequence of $R$-modules splits.*

We get four further equivalent statements by using the "right" versions of (ii) through (v).

## 2 Tor dimension

**2.1**. (Pontryagin dual)

Let $B$ be an $R$-module. The **Pontryagin dual** of $B$ is the right $R$-module

$$B^* := \mathrm{Hom}_{\mathbf{Z}}(B, \mathbf{Q}/\mathbf{Z})$$

with action given by $(fr)(b) = f(rb)$. A similar definition yields a left $R$-module $A^*$ for every right $R$-module $A$.

LEMMA. *Let $A, B$ and $C$ be $R$-modules.*

(i) *If $0 \neq b \in B$, then there exists $f \in B^*$ such that $f(b) \neq 0$.*

(ii) *$A \to B \to C$ is exact if and only if $C^* \to B^* \to A^*$ is exact.*

An $R$-module $A$ is **finitely presented** if there exists an exact sequence $R^n \xrightarrow{\alpha} R^m \to A \to 0$ for some positive integers $m$ and $n$. In this situation, $A$ has the presentation

$$A = \langle e_1, \ldots, e_m \mid \sum_i \alpha_{ij} e_i, 1 \leq j \leq n \rangle,$$

which denotes the free $R$-module on the set of elements to the left of the vertical bar ($|$) modulo the submodule generated by the set of elements to the right (the latter of which is the image of $\alpha$).

Theorem. *Let $A_R$, ${}_RB$, and ${}_RC$ be modules as indicated.*

(i) *The map $\sigma : (A \otimes_R B)^* \to \mathrm{Hom}_R(A, B^*)$ given by $\sigma(f)(a)(b) = f(a \otimes b)$ is an isomorphism that is natural in both $A$ and $B$.*

(ii) *The map $\tau : B^* \otimes_R C \to \mathrm{Hom}_R(C, B)^*$ given by $\tau(f \otimes c)(g) = f(g(c))$ is natural in both $B$ and $C$. It is an isomorphism if $C$ is finitely presented.*

Theorem. *Every finitely presented flat $R$-module is projective.*

Theorem. *Let $B$ be an $R$-module. The following are equivalent:*

(i) *$B$ is flat,*

(ii) *$B^*$ is injective,*

(iii) *the map $I \otimes_R B \to R \otimes_R B$ induced by inclusion is injective for every right ideal $I$ of $R$,*

(iv) $\mathrm{Tor}_1^R(R/I, B) = 0$ *for every right ideal $I$ of $R$.*

**2.2.** (Flat dimension)

Let $B$ be an $R$-module. The **flat dimension** of $B$, written $\mathrm{fd}(B)$, is the least nonnegative integer $d$ for which there exists a flat resolution of $B$ of the form

$$0 \to Q_d \to \cdots \to Q_1 \to Q_0 \to B \to 0,$$

unless there is no such resolution, in which case $\mathrm{fd}(B) = \infty$.

For instance, $B$ is flat if and only if $\mathrm{fd}(B) = 0$. If $R = \mathbf{Z}$, then $\mathrm{fd}(\mathbf{Z}_2) = 1$, since $0 \to 2\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$ is a flat resolution of $\mathbf{Z}_2$ (recalling that projective $\Rightarrow$ flat) and $\mathbf{Z}_2$ is not flat since $\cdot \otimes_{\mathbf{Z}} \mathbf{Z}_2$ does not preserve exactness of $0 \to \mathbf{Z} \to \mathbf{Q}$.

Since a projective resolution is a flat resolution, it follows that $\mathrm{pd}(B) \geq \mathrm{fd}(B)$. One can have strict inequality here since the $\mathbf{Z}$-module $\mathbf{Q}$ is flat (I-10.4) so $\mathrm{fd}(\mathbf{Q}) = 0$, but it is not projective (a nonzero submodule of a free $\mathbf{Z}$-module cannot be divisible) so $\mathrm{pd}(\mathbf{Q}) \neq 0$.

Theorem. *For given $B$ and $d$, the following are equivalent:*

(i) $\mathrm{fd}(B) \leq d$,

(ii) $\mathrm{Tor}_n^R(A, B) = 0$ *for all* $n > d$ *and all* $A$,

(iii) $\mathrm{Tor}_{d+1}^R(A, B) = 0$ *for all* $A$,

(iv) *if* $0 \to M_d \to Q_{d-1} \to \cdots \to Q_1 \to Q_0 \to B \to 0$ *is exact with each* $Q_i$ *flat, then* $M_d$ *is flat.*

When $d = 0$, the equivalence (i) $\Leftrightarrow$ (iii) says that $B$ is flat if and only if $\mathrm{Tor}_1^R(A, B) = 0$ for all $A$, which is a theorem in I-12.3.

There is a similar notion of flat dimension of a right $R$-module $A$ (formally, one could define $\mathrm{fd}(A)$ to be the flat dimension of the (left) $R^{\mathrm{op}}$-module $A$).

**2.3**. (Tor-dimension)

The **Tor dimension** of the ring $R$, written $\mathrm{td}(R)$, is the supremum of the set of flat dimensions of all (left) $R$-modules:

$$\mathrm{td}(R) = \sup\{\mathrm{fd}(B) \mid B \in R\text{-}\mathbf{mod}\}$$

For example, $\mathrm{td}(\mathbf{Z}) = 1$ since $\mathrm{fd}(\mathbf{Z}_2) = 1$ and for any $\mathbf{Z}$-module $A$, there exists an exact sequence $0 \to F_1 \to F_0 \to A \to 0$ with $F_i$ free (hence flat).

THEOREM. *The following numbers are equal:*

(i) $\mathrm{td}(R)$,

(ii) $\sup\{\mathrm{fd}(A) \mid A \in \boldsymbol{mod}\text{-}R\}$,

(iii) $\sup\{\mathrm{fd}(R/I) \mid I$ *is a left ideal of* $R\}$,

(iv) $\sup\{\mathrm{fd}(R/I) \mid I$ *is a right ideal of* $R\}$,

(v) $\sup\{d \mid \mathrm{Tor}_d^R(A, B) \neq 0$ *for some* $A \in \boldsymbol{mod}\text{-}R,\ B \in R\text{-}\boldsymbol{mod}\}$.

Since $\mathrm{fd}(B) \leq \mathrm{pd}(B)$ for an $R$-module $B$, it follows that $\mathrm{td}(R) \leq \mathrm{lgd}(R)$, and, similarly, $\mathrm{td}(R) \leq \mathrm{rgd}(R)$.

THEOREM.

(i) *If* $R$ *is left Noetherian, then* $\mathrm{td}(R) = \mathrm{lgd}(R)$.

(ii) *If $R$ is right Noetherian, then* $\mathrm{td}(R) = \mathrm{rgd}(R)$.

In particular, if $R$ is (left and right) Noetherian, then $\mathrm{lgd}(R) = \mathrm{rgd}(R)$.

We have seen that $R$ is semisimple if and only if its left (or right) global dimension is zero (see theorem in 1.4). The following characterization of semisimple ring using Tor-dimension is therefore a direct consequence of the preceding theorem (and Wedderburn's theorem to see that a semisimple ring is Noetherian).

Corollary. *The ring $R$ is semisimple if and only if it is left Noetherian and* $\mathrm{td}(R) = 0$.

**2.4.** (Von Neumann regular ring)

In this section, we show that the class of rings with zero Tor dimension is precisely the class of von Neumann regular rings.

The ring $R$ is **von Neumann regular** if for each $a \in R$ there exists $x \in R$ such that $axa = a$ (one regards $x$ as a "weak inverse" of $a$). For example, if $R$ is a division ring, then $R$ is von Neumann regular.

*Exercise.* Let $V$ be a vector space over a division ring $D$. Prove that the ring $\mathrm{End}_D(V)$ is von Neumann regular.

The main theorem requires a definition and a lemma. An element $e$ of $R$ is **idempotent** if $e^2 = e$. Both 0 and 1 are idempotent. If $e \in R$ is idempotent, then so is $1 - e$, and we get the left ideal direct sum decomposition $R = Re \dot{+} R(1-e)$ (which is a proper decomposition if $e \neq 0, 1$). Conversely, if $R = I \dot{+} J$ with $I$ and $J$ left ideals of $R$ and one writes $1 = e + f$ with $e \in I$ and $f \in J$, then $e$ and $f$ are idempotent. (Indeed, $e = e1 = e^2 + ef$ and since $e, e^2 \in I$ and $ef \in J$, we get $ef = 0$ so that $e^2 = e$. Similarly for $f$.)

Lemma. *Assume that $R$ is von Neumann regular. Let $I$ be a finitely generated left ideal of $R$.*

(i) *$I = Re$ for some idempotent element $e$ of $R$, so that $R = I \dot{+} R(1-e)$.*

(ii) *$I$ and $R/I$ are projective $R$-modules.*

Theorem. *The following are equivalent:*

(i) *$R$ is von Neumann regular,*

(ii) $\mathrm{td}(R) = 0$,

(iii) *every $R$-module is flat,*

(iv) *$R/I$ is projective for every finitely generated left ideal $I$ of $R$.*

Corollary. *$R$ is semisimple if and only if it is von Neumann regular and left Noetherian.*

$\mathrm{End}_D(V)$ (as in the exercise above) is von Neumann regular, but it is not semisimple if $V$ is infinite dimensional (by Wedderburn's theorem, for example). This example also shows that one can have $\mathrm{lgd}(R)$ strictly greater than $\mathrm{td}(R)$.

# 3 Spectral sequence

**3.1**. (Definition)

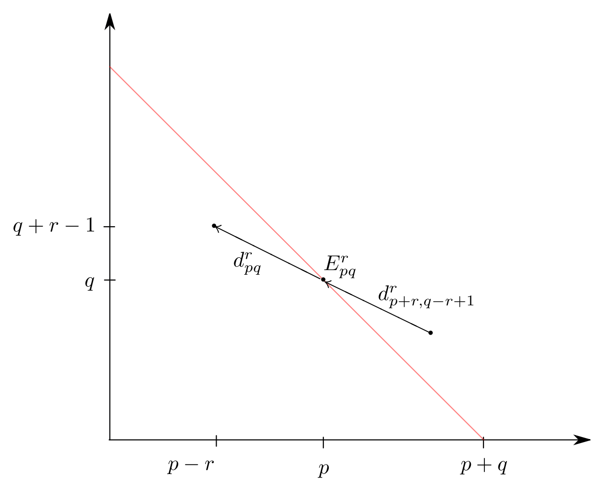
A **(homology) spectral sequence** $E$ consists of the following:

(i) for each integer $r \geq 0$ a doubly indexed family of $R$-modules $\{E^r_{pq}\}$ $(p, q \in \mathbf{Z})$;

(ii) $R$-homomorphisms $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ such that $d^r_{p-r,q+r-1} d^r_{pq} = 0$ for all $p, q, r$;

(iii) isomorphisms $E^{r+1}_{pq} \to \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1}$ for each $p, q, r$.

(A **cohomology spectral sequence** is defined similarly, except with arrows reversed.)

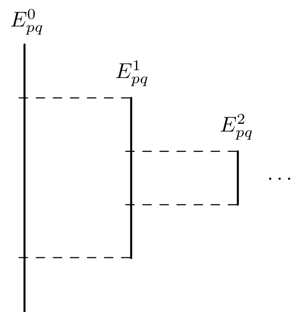



For each $r$, the family $\{E^r_{pq}\}$ is the $r$**th sheet** of the sequence. It is convenient to locate the term $E^r_{pq}$ of the $r$th sheet at the point $(p, q)$ of the plane (see figure).

The maps $d^r_{pq}$ are **differentials**. They point downward when $r = 0$, to the left when $r = 1$, and up and to the left for $r > 1$ (as in the figure). Their lengths increase with $r$ (for $r > 0$).

The **total degree** of the term $E^r_{pq}$ is $p + q$. Terms of identical total degree lie along lines of slope $-1$ (red).

The differentials form chain complexes along the lines with slope $-(r-1)/r$. The $r$th sheet consists of the homology modules of the complexes in the $(r-1)$st sheet. In particular, each term $E^r_{pq}$ is a subquotient (i.e., a quotient of a submodule) of the corresponding term on the preceding sheet:




In particular, if any term $E^r_{pq}$ is zero, then the corresponding terms on the subsequent sheets are also zero.

If the modules $E^0_{pq}$ on the initial sheet are zero whenever $p < 0$ or $q < 0$, then $E$ is a **first quadrant spectral sequence**. In this case, for each pair $(p, q)$, there is a large enough $r$ such that the differential leaving $E^r_{pq}$ lands in the second quadrant and the differential arriving at $E^r_{pq}$ comes from the fourth quadrant, so that

$$E^r_{pq} = E^{r+1}_{pq} = E^{r+2}_{pq} = \cdots .$$

This module is denoted $E^\infty_{pq}$.

More generally, the spectral sequence $E$ is **bounded** if, for each integer $n$, there exist only finitely many nonzero terms $E^0_{pq}$ of total degree $n$ (e.g., a first quadrant spectral sequence is bounded). Since differentials decrease total degree by one and increase in length with $r$, a bounded spectral sequence has the property that for each pair $(p, q)$, the terms $E^r_{pq}$ eventually stabilize as above, so that $E^\infty_{pq}$ is defined.

The bounded spectral sequence $E$ **converges** to the family $\{H_n\}$ of $R$-modules if for each $n$ there exists a filtration

$$\cdots F_{p-1}H_n \leq F_pH_n \leq F_{p+1}H_n \cdots$$

of $H_n$ with $F_sH_n = 0$ and $F_tH_n = H_n$ for some $s \leq t$ and, for each pair $(p, q)$ there exists an isomorphism

$$\beta_{pq} : E^\infty_{pq} \to F_pH_{p+q}/F_{p-1}H_{p+q}.$$

In this case, we write

$$E^0_{pq} \Rightarrow H_{p+q}.$$

**Example**. (Convergent first quadrant spectral sequence)

Let $E$ be a first quadrant spectral sequence and assume that $E^0_{pq} \Rightarrow H_{p+q}$. Fix a nonnegative integer $n$. According to the definition of convergence, $H_n$ has a descending sequence of submodules with successive quotients that can be read off by going to a sheet containing each $E^\infty_{pq}$ with $p+q=n$ (such a sheet exists), starting on the $x$-axis at position $n$ and proceeding northwest to get

$$E^\infty_{n0}, E^\infty_{n-1,1}, E^\infty_{n-2,2}, \ldots, E^\infty_{0,n}.$$

**3.2**. (Spectral sequence of a filtered complex)

Let $C$ be a chain complex of $R$-modules. For each $n$, let $C'_n$ be a submodule of $C_n$ and assume that $d_n(C'_n) \subseteq C'_{n-1}$ for each $n$. Then $C'$ is a chain complex with differentials induced by the differentials $d_n$; it is called a **subcomplex** of $C$, written $C' \leq C$ (or $C \geq C'$).

A **filtration** of $C$ is a family $\{F_p(C)\}_{p\in\mathbf{Z}}$ of subcomplexes of $C$ with $F_p(C) \geq F_{p-1}(C)$ for each $p$. Such a filtration is **bounded** if for each $n$ there exist $s \leq t$ such that $F_s(C) = 0$ and $F_t(C) = C$.

THEOREM. *Let $\{F_p(C)\}$ be a bounded filtration of the chain complex $C$. There exists a spectral sequence $E$ with differentials induced by the differentials of $C$ such that*

$$E^0_{pq} = F_p(C_{p+q})/F_{p-1}(C_{p+q}) \Rightarrow H_{p+q}(C).$$

**3.3**. (Spectral sequence of a bicomplex)

A **bicomplex** $C$ is a doubly-indexed family $\{C_{pq}\}_{p,q\in\mathbf{Z}}$ of $R$-modules together with maps $d^h_{pq} : C_{pq} \to C_{p-1,q}$ and $d^v_{pq} : C_{pq} \to C_{p,q-1}$ such that

$$d^h d^h = 0, \quad d^v d^v = 0, \quad d^v d^h + d^h d^v = 0.$$

The module $C_{pq}$ is visualized as being located at the point $(p,q)$ of the plane. The maps $d^h$ are horizontal and point to the left, while the maps $d^v$ are vertical and point down. The last condition says that the squares are *anti*commutative: $d^v d^h = -d^h d^v$.

The **total complex** $\mathrm{Tot}(C)$ of the bicomplex $C$ is the chain complex with $n$th term

$$\mathrm{Tot}(C)_n = \coprod_{p+q=n} C_{pq}$$

and with differential $d_n$ induced by the map on $C_{pq}$ given by $d^h + d^v$. One checks using the properties of $d^h$ and $d^v$ that $d_n^2 = 0$ as required.

There are two natural filtrations associated with the total complex, given by

$${}^I F_p(\mathrm{Tot}(C))_n = \coprod_{i \le p} C_{i,n-i}$$

and

$${}^{II} F_p(\mathrm{Tot}(C))_n = \coprod_{i \le p} C_{n-i,i}.$$

We give a description of the first few sheets of the corresponding spectral sequences, denoted ${}^I E$ and ${}^{II} E$, respectively:

$${}^I E^0_{pq} = {}^I F_p(\mathrm{Tot}(C))_n / {}^I F_{p-1}(\mathrm{Tot}(C))_n \cong C_{pq} \quad (n = p+q).$$

The differential $d^0_{pq}$ is induced by the differential $d_n = d^h_{pq} + d^v_{pq}$, and since $d^h_{pq}$ maps into ${}^I F_{p-1}(\mathrm{Tot}(C))_n$ it follows that $d^0_{pq} = d^v_{pq}$ (identifying ${}^I E^0_{pq}$ with $C_{pq}$). Therefore, the next sheet of the spectral sequence is obtained by taking homology of the bicomplex $C$ in the vertical direction, which is what the following notation signifies:

$${}^I E^1_{pq} = H^v_{pq}(C) = \ker d^v_{pq} / \operatorname{im} d^v_{p,q+1}.$$

Again, the differential $d^1_{pq}$ is induced by the differential $d_n = d^h_{pq} + d^v_{pq}$, but now the map induced by $d^v_{pq}$ is zero, so we have $d^1_{pq} = \bar{d}^h_{pq}$ where the bar denotes the induced map. Therefore, the next sheet of the spectral sequence is obtained by taking homology relative to maps induced by the horizontal maps of the complex $C$, which again we use suggestive notation for:

$${}^I E^2_{pq} = H^h_{pq}(H^v(C)) = \ker \bar{d}^h_{pq} / \operatorname{im} \bar{d}^h_{p+1,q}.$$

The description of the first few sheets of ${}^{II} E$ is very similar, except that there is a reversal of indices that occurs:

$$\begin{aligned}
{}^{II}E^0_{pq} &= {}^{II}F_p(\mathrm{Tot}(C))_n/{}^{II}F_{p-1}(\mathrm{Tot}(C))_n \cong C_{qp} \quad (n = p+q),\\
{}^{II}E^1_{pq} &= H^h_{qp}(C) \quad \text{(homology of } C \text{ using horizontal maps)},\\
{}^{II}E^2_{pq} &= H^v_{qp}(H^h(C)) \quad \text{(homology of sheet 1 using maps induced by}\\
&\qquad\qquad \text{vertical maps of } C\text{)}.
\end{aligned}$$

According to the previous section, both of these spectral sequences converge to the homology of the total complex if the filtrations are bounded, which is the case if the bicomplex $C$ is bounded (same definition as for bounded spectral sequence).

Theorem. *If the bicomplex $C$ is bounded, then*

$${}^{I}E^2_{pq} = H^h_{pq}(H^v(C)) \Rightarrow H_n(\mathrm{Tot}(C)),$$

*and*

$${}^{II}E^2_{pq} = H^v_{qp}(H^h(C)) \Rightarrow H_n(\mathrm{Tot}(C)),$$

*where $n = p + q$.*

The meaning of the notation is that the spectral sequence ${}^{I}E$ converges to $H_n(\mathrm{Tot}(C))$ and its second sheet $(p, q)$ term is as indicated. Similarly for ${}^{II}E$.

**3.4**. (Sign trick)

Let $\{C_{pq}\}_{p,q\in\mathbf{Z}}$ be a doubly indexed family of $R$-modules and let $\delta^h_{pq} : C_{pq} \to C_{p-1,q}$ and $\delta^v_{pq} : C_{pq} \to C_{p,q-1}$ be maps such that

$$\delta^h\delta^h = 0, \qquad \delta^v\delta^v = 0, \qquad \delta^v\delta^h = \delta^h\delta^v$$

(i.e., $C$ is a chain complex of chain complexes). The family $C$ becomes a bicomplex if one uses the following **sign trick** to define the maps:

$$d^h_{pq} = \delta^h_{pq}, \qquad d^v_{pq} = (-1)^p\delta^v_{pq}.$$

(The sign $(-1)^p$ changes the squares from commutative to anticommutative.)

# 4 Applications of spectral sequence

**4.1**. (Balancing Tor)

We defined the functor $\mathrm{Tor}_n^R(A, \cdot)$ to be $L_n(A \otimes_R \cdot)$, so that

$$\mathrm{Tor}_n^R(A, B) = H_n(A \otimes_R Q)$$

where $Q \to B \to 0$ is a projective resolution of $B$. We stated that we could just as well have defined

$$\mathrm{Tor}_n^R(A, B) = H_n(P \otimes_R B)$$

where $P \to A \to 0$ is a projective resolution of $A$. Our first application of a spectral sequence is the proof of the equivalence of these definitions.

THEOREM. *With the notation above,* $H_n(A \otimes_R Q) \cong H_n(P \otimes_R B)$.

*Proof.* (Sketch) Put $C_{pq} = P_p \otimes_R Q_q$ and use the sign trick to view $C$ as a bicomplex. Using the previous section, we find that

$${}^I E^2_{pq} \cong \begin{cases} H_n(P \otimes_R B) & q = 0, \\ 0 & q \neq 0 \end{cases}$$

$(n = p + q)$. In particular, ${}^I E^\infty_{pq} = {}^I E^2_{pq}$ so that $\mathrm{Tot}(C)_n \cong H_n(P \otimes_R B)$. Similarly, using ${}^{II}E$, we find that $\mathrm{Tot}(C)_n \cong H_n(A \otimes_R Q)$ and the claim follows. □

A similar proof, using a cohomology spectral sequence, shows that the two ways to compute $\mathrm{Ext}^n_R(A, B)$ agree.

**4.2**. (Künneth spectral sequence)

Let $P$ be a chain complex of flat right $R$-modules that is bounded below (i.e., there exists an integer $N$ such that $P_n = 0$ for all $n < N$). Let $M$ be an $R$-module. The following result relates the homology modules of the complex $P \otimes_R M$ to those of the complex $P$.

THEOREM. *There exists a bounded spectral sequence $E$ such that*

$$E^2_{pq} = \mathrm{Tor}_p(H_q(P), M) \Rightarrow H_n(P \otimes_R M),$$

*where* $n = p + q$.

$E$ is called the **Künneth spectral sequence**.

Corollary. *Let $P$ be a chain complex of flat right $R$-modules and assume that $\operatorname{im} d_n$ is flat for each $n$. For each $n$ there exists an exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes M) \to \operatorname{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

This result is known as the **universal coefficient theorem of homology**. There is a corresponding result involving $\operatorname{Ext}^1$, which is proved using a cohomology analog of the Künneth spectral sequence. Both results have generalizations referred to as the "Künneth formulas."

Note that if $M$ is flat, then the result says that $H_n(P \otimes M) \cong H_n(P) \otimes_R M$ (though this statement is actually required in the proof so that an independent argument is required).

**4.3.** (Change of rings)

Let $\varphi : R \to S$ be a ring homomorphism. We use this homomorphism to view any $S$-module $B$ as an $R$-module: $rb = \varphi(r)b \quad (r \in R, b \in B)$.

Let $A$ be a right $R$-module. Since $S$ is an $(R, S)$-bimodule, the tensor product $A \otimes_R S$ is a right $S$-module with action $(a \otimes s)s' = a \otimes (ss')$ $(a \in A, s, s' \in S)$. If $P$ is a chain complex, then each term of the complex $P \otimes_R S$ is a right $S$-module and the differentials are $S$-homomorphisms. Therefore, $\operatorname{Tor}_q^R(A, S)$ is a right $S$-module for each $q$.

Theorem. *Let $\varphi : R \to S$ be a ring homomorphism and let $A_R$ and ${}_SB$ be modules as indicated. There exists a first quadrant spectral sequence $E$ such that*

$$E^2_{pq} = \operatorname{Tor}_p^S(\operatorname{Tor}_q^R(A, S), B) \Rightarrow \operatorname{Tor}_n^R(A, B),$$

*where $n = p + q$.*

There is a corresponding theorem involving Ext.

Corollary. *Let $\varphi : R \to S$ be a ring homomorphism. For any $S$-module $B$ we have*

$$\operatorname{fd}_R(B) \leq \operatorname{fd}_S(B) + \operatorname{fd}_R(S).$$

(The subscripts indicate the rings to be used in the definition of flat dimension.)

# 5 Cohomology of groups

**5.1**. (Group extension problem)

Let $G$ and $A$ be groups. An **extension** of $G$ by $A$ is a group $E$ having a normal subgroup isomorphic to $A$ with corresponding quotient isomorphic to $G$.

An important unsolved problem is the classification of all such extensions for arbitrary finite groups $G$ and $A$. If this problem were solved, then in a sense one would know all finite groups. In fact, for this, one would need only classify all such extensions with $A$ simple. Indeed, arguing by induction on the group order, a given nontrivial finite group $E$ has a simple normal subgroup (known by the Classification Theorem) with corresponding quotient of order less than $|E|$ (hence known by the induction hypothesis), so the group $E$ is known by the assumed solution to the extension problem.

We show that, under the assumption that $A$ is abelian, the essentially distinct extensions of $G$ by $A$ are in one-to-one correspondence with a certain Ext group. This leaves (for the abelian $A$ case) the problem of computing that Ext group, and for this standard techniques in homological algebra can be applied.

**5.2**. (Cohomology groups)

Let $G$ be a group. A $G$**-module** is an abelian group $A$ together with a map $G \times A \to A$, written $(g, a) \mapsto g \cdot a$, such that for all $g, h \in G$ and $a, b \in A$

(i) $1 \cdot a = a$ (1 is the identity element of $G$),

(ii) $(gh) \cdot a = g \cdot (h \cdot a)$, and

(iii) $g \cdot (a + b) = g \cdot a + g \cdot b$.

(Properties (i) and (ii) say that the map gives an **action** of $G$ on $A$ making $A$ a $G$**-set**.)

A $G$-module $A$ gives rise to a homomorphism $\varphi : G \to \mathrm{End}(A)$ given by $\varphi(g)(a) = g \cdot a$. Conversely, such a homomorphism gives rise to a $G$-module structure on $A$ by putting $g \cdot a = \varphi(g)(a)$.

A $G$-module $A$ is **trivial** if $g \cdot a = a$ for all $g \in G$ and $a \in A$. Any abelian group can be viewed as a trivial $G$-module by using this action.

The $G$-module $A$ is trivial if and only if the corresponding homomorphism $\varphi : G \to \mathrm{End}(A)$ is the trivial homomorphism.

For us, the main example of a $G$-module arises in connection with an extension of $G$. Let $E$ be an extension of $G$ by the abelian group $A$, so that $A$ is identified with a normal subgroup of $E$ and $G$ is identified with $E/A$. For $g = eA \in G$ and $a \in A$, we put

$$g \cdot a = {}^{e}a := eae^{-1}.$$

One shows that the action is independent of the chosen coset representative (and using this fact, we write unambiguously ${}^{g}a$ for ${}^{e}a$). The resulting $G$-module $A$ is trivial if and only if $A$ is contained in the center of $E$. In this case $E$ is a **central extension** of $G$ by $A$.

We use the new language to reformulate the statement of the extension problem. Let $A$ be a $G$-module. An extension $E$ of $G$ by $A$ corresponds to an exact sequence of groups

$$0 \to A \to E \to G \to 1. \qquad (*)$$

(We write 0 on the left since we are viewing $A$ as an additive group.) Two extensions $E$ and $E'$ of $G$ by $A$ are **equivalent** if there exists a homomorphism $\varphi : E \to E'$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

In this case, the five lemma shows that $\varphi$ is in fact an isomorphism.

The extension (*) **respects the $G$-module structure of** $A$ if $g \cdot a = {}^{g}a$ for each $g \in G$ and $a \in A$. A solution to the extension problem would be a determination of the set $e(G, A)$ of equivalence classes of those extensions of $G$ by $A$ that respect the $G$-module structure of $A$.

Let $\mathbf{Z}G$ be the group ring of $G$ over $\mathbf{Z}$. Thus $\mathbf{Z}G$ is the free abelian group on $G$ with multiplication induced by the group operation of $G$ so that

$$\Big(\sum_{g \in G} \alpha_g g\Big)\Big(\sum_{h \in G} \beta_h h\Big) = \sum_{g,h \in G} \alpha_g \beta_h (gh).$$

If $A$ is a $G$-module, then the action of $G$ on $A$ extends linearly to give a $\mathbf{Z}G$-module structure on $A$:

$$\left(\sum_{g\in G}\alpha_g g\right)a = \sum_{g\in G}\alpha_g g\cdot a.$$

Conversely, any $\mathbf{Z}G$-module can be viewed as a $G$-module by restricting the scalars from $\mathbf{Z}G$ to $G$. If $A$ is a trivial $G$-module, then $(\sum \alpha_g g)a = (\sum \alpha_g)a$ ($\alpha_g \in \mathbf{Z}$, $a \in A$).

Let $A$ be a $G$-module. For a nonnegative integer $n$, the **$n$th cohomology group of $G$ with coefficients in $A$** is

$$H^n(G,A) := \operatorname{Ext}^n_{\mathbf{Z}G}(\mathbf{Z},A),$$

where $\mathbf{Z}$ is viewed as a trivial $G$-module. We will show that $e(G,A)$ is in one-to-one correspondence with the set $H^2(G,A)$. We will also give interpretations of $H^0(G,A)$ and $H^1(G,A)$.

**5.3.** (Standard resolution)

To facilitate the interpretations of $H^n(G,A)$, we introduce a particular free resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$.

Let $B_n$ be the free $\mathbf{Z}G$-module on the set $\{[\,g_1\,|\,g_2\,|\cdots|\,g_n\,]\mid 1\neq g_i\in G\}$. Extend the meaning of the symbol $[\,g_1\,|\,g_2\,|\cdots|\,g_n\,]$ by defining it to be 0 if $g_i = 1$ for some $i$.

By convention, $B_0$ is the free $\mathbf{Z}G$-module on the singleton set $\{[\ ]\}$ and therefore identifies with $\mathbf{Z}G$. The map $d_0 : B_0 \to \mathbf{Z}$ given by $\sum \alpha_g g \mapsto \sum \alpha_g$ is a $\mathbf{Z}G$-epimorphism called the **augmentation map of $\mathbf{Z}G$**.

For $n > 0$, let $d_n : B_n \to B_{n-1}$ be the $\mathbf{Z}G$-homomorphism determined by the formula

$$\begin{aligned} d_n([\,g_1\,|\,g_2\,|\cdots|\,g_n\,]) &= g_1[\,g_2\,|\cdots|\,g_n\,] \\ &\quad + \sum_{i=1}^{n-1}(-1)^i[\,g_1\,|\cdots|\,g_ig_{i+1}\,|\cdots|\,g_n\,] \\ &\quad + (-1)^n\,[\,g_1\,|\cdots|\,g_{n-1}\,]. \end{aligned}$$

THEOREM. *The chain complex $B \to \mathbf{Z} \to 0$ is a free resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$.*

This is the **standard resolution** of **Z**. (It is also known as the **bar resolution** of **Z**, presumably due to the bars | in the notation of the generators of $B_n$.)

Let $A$ be a $G$-module. Using the standard resolution, we have that $H^n(G, A)$ is the $n$th cohomology group of the chain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}G}(B_0, A) \xrightarrow{d_1{}^*} \mathrm{Hom}_{\mathbf{Z}G}(B_1, A) \xrightarrow{d_2{}^*} \mathrm{Hom}_{\mathbf{Z}G}(B_2, A) \xrightarrow{d_3{}^*} \cdots$$

The group $\mathrm{Hom}_{\mathbf{Z}G}(B_n, A)$ identifies with the group $C^n(G, A)$ of all functions $f : G^n \to A$ such that $f(g_1, \ldots, g_n) = 0$ if $g_i = 1$ for some $i$, where $G^n = G \times \cdots \times G$ ($n$ factors). (We view $G^0$ as a singleton set and for $f \in C^0(G, A)$ we write $f() \in A$ for the image of the single element.) The elements of $C^n(G, A)$ are $n$-**cochains**. For $f \in C^n(G, A)$, we have

$$\begin{aligned} d_{n+1}{}^*(f)(g_0, g_1, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) & \\ + \sum_{i=0}^{n} (-1)^{i+1} f(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) & \\ + (-1)^{n+1} f(g_0, \ldots, g_{n-1}) & \end{aligned}$$

The elements of $Z^n(G, A) := \ker d_{n+1}{}^*$ are $n$-**cocycles** and the elements of $B^n(G, A) := \mathrm{im}\, d_n{}^*$ are $n$-**coboundaries**. By definition, $H^n(G, A) = Z^n(G, A)/B^n(G, A)$.

**5.4.** (Extension and $H^2(G, A)$)

Let $A$ be a $G$-module and let

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$$

be an extension of $G$ by $A$ that respects the $G$-module structure of $A$. Let $\sigma : G \to E$ be a section of $\pi$, that is, $\pi\sigma = 1_G$ ($\sigma$ need not be a homomorphism). Assume that $\sigma(1) = 1$.

For $g, h \in G$, we have

$$\pi(\sigma(g)\sigma(h)) = gh = \pi(\sigma(gh))$$

so that

$$\sigma(g)\sigma(h) = [g, h]\sigma(g)\sigma(h)$$

for some $[g, h] \in A$ (we have identified $A$ with its image). The function $[\,,\,] = [\,,\,]_{E,\sigma} : G \times G \to A$ is the **factor set** of the extension relative to $\sigma$.

LEMMA. *A function* $[\,,\,] : G \times G \to A$ *is a factor set (of some extension relative to some section) if and only if it is a 2-cocycle, that is, if and only if for all* $g, h, k \in G$*,*

(i) $[g, 1] = 0$ *and* $[1, g] = 0$*,*

(ii) $g \cdot [h, k] - [gh, k] + [g, hk] - [g, h] = 0$.

Recall that $e(G, A)$ is the set of equivalence classes of those extensions of $G$ by $A$ that respect the $G$-module structure of $A$.

THEOREM. *There is a one-to-one correspondence* $e(G, A) \leftrightarrow H^2(G, A)$*.*

A bijection $\Phi : e(G, A) \leftrightarrow H^2(G, A)$ is given by $\Phi(\bar{E}) = \overline{[\,,\,]}_{E,\sigma}$ (any $\sigma$), where the bars denote classes. The lemma shows that this function maps into $H^2(G, A)$ and is surjective, so one just needs to check that it is well-defined and injective.

**5.5.** Schur-Zassenhaus Theorem

Let $G$ and $A$ be groups and let $\varphi : G \to \mathrm{Aut}(A)$, $g \mapsto \varphi_g$, be a homomorphism (called an **action** of the group $G$ on the group $A$). The set $A \times G$ is a group with product given by

$$(a, g)(b, h) = (a\varphi_g(b), gh).$$

(The identity element is $(1, 1)$ and the inverse of $(a, g)$ is $(\varphi_{g^{-1}}(a^{-1}), g^{-1})$.) This group is the **semidirect product** of $A$ and $G$ with respect to the action $\varphi$, written $A \rtimes_\varphi G$. (When $\varphi$ is trivial, this is just the direct product.)

The usual injections identify $A$ and $G$ with subgroups of the semidirect product $E = A \rtimes_\varphi G$. . Moreover, $A \lhd E$ (which is what the notation is meant to indicate) and

(i) $AG = E$,

(ii) $A \cap G = 1$.

Conversely, if $E$ is a group with a subgroup $A$ and a normal subgroup $G$ satisfying these two properties, then $E$ is the **(internal) semidirect**

**product** of $A$ and $G$, written $A \rtimes G$. In this case, $\varphi : G \to \mathrm{Aut}(A)$ given by $\varphi_g(a) = gag^{-1}$ is an action of $G$ on $A$ and $E$ is isomorphic to $A \rtimes_\varphi G$.

Let $G$ and $A$ be groups. An extension

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$$

of $G$ by $A$ is **split** if there exists a homomorphism $\sigma : G \to E$ such that $\pi\sigma = 1_G$. In this case $\sigma$ is an injection and we use it to identify $G$ with its image in $E$. Identifying $A$ with its image as well, we then have $E = AG$ and $A \cap G = 1$ so that $E = A \rtimes G$.

THEOREM (Schur-Zassenhaus). *If $G$ and $A$ are finite groups with $(|G|, |A|) = 1$, then every extension of $G$ by $A$ is split.*

The proof depends on the proof of the special case where $A$ is abelian, which we now sketch. Let $G$ and $A$ be finite groups with $A$ abelian.

LEMMA. *If $A$ is a $G$-module, then*

(i) $|G|\, H^n(G, A) = 0$ *($n \neq 0$),*

(ii) $|A|\, H^n(G, A) = 0$.

Let $E$ be an extension of $G$ by $A$ and assume that $(|G|, |A|) = 1$. The extension induces a $G$-module structure on $A$. Since $a|G| + b|A| = 1$ for some integers $a$ and $b$, it follows from the lemma that $H^2(G, A) = 0$. By the theorem of 5.4, $E$ is equivalent to an extension $E'$ with trivial factor set relative to a section $\sigma$. The definition of factor set shows that $\sigma$ is a homomorphism so that the extension $E'$ splits. Since any extension equivalent to a split extension is also split, the special case of the Schur-Zassenhaus theorem follows.

**5.6.** (Fixed point set and $H^0(G, A)$)

Let $G$ be a group and let $A$ be a $G$-module. The **fixed point set** of $A$ is

$$A^G := \{a \in A \mid g \cdot a = a \text{ for all } g \in G\}$$

It is a $G$-submodule of $A$. In fact it is the unique maximal trivial $G$-submodule of $A$.

The **fixed point functor** $\cdot^G$ is the functor from the category of $G$-modules to the category of abelian groups (or trivial $G$-modules) that sends a $G$-module $A$ to the abelian group $A^G$ and sends a $G$-homomorphism $f : A \to B$ to the restriction $f|_{A^G}$.

The map $\eta_A : \mathrm{Hom}_{\mathbf{Z}G}(Z, A) \to A^G$ given by $\eta_A(f) = f(1)$ defines an equivalence of functors $\mathrm{Hom}_{\mathbf{Z}G}(Z, \cdot\,) \simeq \cdot^G$. This provides an interpretation of the cohomology group $H^0(G, A)$.

Corollary. $H^0(G, A) \cong A^G$.

**5.7.** (Automorphisms of extensions and $H^1(G, A)$)

Let $G$ be a group and let $A$ be a $G$-module. A function $\delta : G \to A$ is a 1-cocycle (that is, an element of $Z^1(G, A)$) if and only if

$$\delta(gh) = g \cdot \delta(h) + \delta(g)$$

for each $g, h \in G$ (see 5.3 and note that the requirement $\delta(1) = 0$ follows from this formula), and it is a 1-coboundary (that is, an element of $B^1(G, A)$) if and only if there exists some $a \in A$ such that

$$\delta(g) = g \cdot a - a$$

for all $g \in G$ (see 5.3 and note that we have written $a$ for $f()$).

Often, a 1-cocycle is referred to as a **derivation** (or **crossed homomorphism**) from $G$ to $A$, and a 1-coboundary is then called an **inner derivation** (or **principal crossed homomorphism**). With this terminology, the cohomology group $H^1(G, A) = Z^1(G, A)/B^1(G, A)$ can be described as the group of derivations from $G$ to $A$ modulo the group of inner derivations.

Let

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$$

be an extension of $G$ by $A$ that respects the $G$-module structure on $A$. Denote by $\mathrm{Aut}'(E)$ the group of those automorphisms $\varphi$ of $E$ for which the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Let $\mathrm{Inn}'(E)$ denote the set $\{\iota_a \mid a \in A\}$, where $\iota_a$ is the inner automorphism of $E$ given by $\iota_a(e) = a^{-1}ea$. Then $\mathrm{Inn}'(E)$ is a normal subgroup of $\mathrm{Aut}'(E)$ (a consequence of the theorem below). Let $\overline{\mathrm{Aut}'(E)}$ denote the corresponding quotient.

Let $\varphi \in \mathrm{Aut}'(E)$ and let $\sigma : G \to E$ be a section of $\pi$ (so that $\pi\sigma = 1$) and assume that $\sigma(1) = 1$. For each $g \in G$ there exists a unique $\delta(g) \in A$ such that

$$\varphi(\sigma(g)) = \delta(g)\sigma(g).$$

This defines a function $\delta = \delta_\varphi : G \to A$ with $\delta(1) = 0$ (the operation in $A$ is written using additive notation).

Theorem. *The function $\Phi : \mathrm{Aut}'(E) \to Z^1(G, A)$ given by $\Phi(\varphi) = \delta_\varphi$ is an isomorphism. Under this isomorphism, $\mathrm{Inn}'(E)$ corresponds to $B^1(G, A)$. In particular,*

$$H^1(G, A) \cong \overline{\mathrm{Aut}'(E)}.$$

If the section $\sigma$ is a *homomorphism* (so that the extension is split), then $G_1 = \sigma(G)$ is called a **complement** of $A$ in $E$. Viewing $A$ as a subgroup of $E$, we then have $AG_1 = E$ and $A \cap G_1 = 1$.

Corollary. *If $G_1$ and $G_2$ are two complements of $A$ in $E$, then there exists $\varphi \in \mathrm{Aut}'(E)$ such that $\varphi(G_1) = G_2$.*

Assume that $G$ and $A$ are finite with $(|G|, |A|) = 1$. The Schur-Zassenhaus theorem (5.5) says that the extension $E$ splits so that there exists a complement $G_1$ of $A$ in $E$. Let $G_2$ be another complement. By the corollary, there exists $\varphi \in \mathrm{Aut}'(E)$ such that $\varphi(G_1) = G_2$. But the lemma of 5.4 gives $H^1(G, A) = 0$ so the theorem above implies that $\varphi = \iota_a$ for some $a \in A$, whence $G_2 = a^{-1}G_1a$.

Corollary. *In the statement of the Schur-Zassenhaus theorem, any two complements of $A$ in $E$ are conjugate.*

Our proof required that $A$ be abelian, but the corollary is true in general.

# 6 Derived category

**6.1**. (Introduction)

Associated to an abelian category $\mathcal{A}$ is a certain category $\mathbf{D}(\mathcal{A})$ called the "derived category of $\mathcal{A}$." This new category is designed for doing the homological algebra of $\mathcal{A}$ in that it simplifies the notions of derived functors and spectral sequences.

Here is a quick sketch of the construction. The objects of $\mathbf{D}(\mathcal{A})$ are simply the cochain complexes of $\mathcal{A}$. The morphisms are obtained from chain maps in a two-step process. First, one mods out null homotopic chain maps, which has the effect of making two chain maps equal when they were merely homotopic before (this produces the "homotopy category" $\mathbf{K}(\mathcal{A})$). Next, one localizes at those chain maps that induce isomorphisms on homology (the so-called "quasi-isomorphisms"). This has the effect of making quasi-isomorphisms into actual isomorphisms.

As an illustration of the suitability of $\mathbf{D}(\mathcal{A})$ for doing homological algebra, we have the following (assuming $\mathcal{A}$ has enough injectives): For any objects $A$ and $B$ of $\mathcal{A}$,

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[n]),$$

where on the right, $A$ is the complex with $A$ as degree zero term and zeros elsewhere and $B[n]$ is the complex with $B$ as degree $-n$ term and zeros elsewhere.

The homotopy category $\mathbf{K}(\mathcal{A})$ is an example of a "triangulated category" (and in turn the derived category $\mathbf{D}(\mathcal{A})$ is as well). The construction process makes use of this additional structure.

**6.2**. (Shift functor)

We fix an abelian category $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{A})$ be the category of chain complexes in $\mathcal{A}$, and let $A \in \mathrm{ob}\,\mathbf{Ch}(\mathcal{A})$. We put $A^n = A_{-n}$ and $d^n = d_{-n}$ and call the resulting complex

$$\cdots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots$$

a **cochain complex**. (Derived categories were originally developed to deal with cohomology and the notation best suited for that theory has persisted.)

For an integer $s$, let $A[s]$ be the cochain complex with

$$A[s]^n = A^{n+s} \qquad d^n_{A[s]} = (-1)^s d^{n+s}_A,$$

and for a chain map $f : A \to B$ define $f[s] : A[s] \to B[s]$ by $f[s]^n = f^{n+s}$.

Then $[s]$ is a functor from $\mathbf{Ch}(\mathcal{A})$ to itself, called the **shift functor** of degree $s$.

**6.3.** (Homotopy category $\mathbf{K}(\mathcal{A})$)

Let $A$ and $B$ be two cochain complexes. We recall some definitions. A chain map $f : A \to B$ is **null homotopic** (written $f \sim 0$) if there exist morphisms $s^n : A^n \to B^{n-1}$ such that $s^{n+1}d_A^n + d_B^{n-1}s^n = f^n$ for all $n$. Two chain maps $f, g : A \to B$ are **homotopic** (written $f \sim g$) if $f - g \sim 0$. The set of null homotopic chain maps in $\mathrm{Hom}(A, B)$ is a subgroup. Write $\overline{\mathrm{Hom}(A, B)}$ for the corresponding quotient.

Let $f$, $g$, and $h$ be chain maps with $g \sim 0$. If $gf$ is defined, then $gf \sim 0$. Similarly, if $hg$ is defined, then $hg \sim 0$. It follows that one obtains a well-defined composition $\overline{\mathrm{Hom}(B, C)} \times \overline{\mathrm{Hom}(A, B)} \to \overline{\mathrm{Hom}(A, C)}$ by putting $\bar{g}\bar{f} = \overline{gf}$.

The **homotopy category** $\mathbf{K}(\mathcal{A})$ of $\mathcal{A}$ is the category with cochain complexes as objects, with morphisms $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \overline{\mathrm{Hom}(A, B)}$, and with composition as just defined.

If $f, g : A \to B$ are chain maps and $f \sim g$, then $H^n(f) = H^n(g)$ for each $n$. Therefore, we get a well-defined functor $H^n : \mathbf{K}(\mathcal{A}) \to \mathbf{Ab}$ by putting $H^n(\bar{f}) = H^n(f)$.

We will often write $\bar{f}$ as just $f$ and rely on phrases such as $f = g$ *in* $\boldsymbol{K}(\mathcal{A})$ (meaning $f \sim g$) when clarification is required.

**6.4.** (Trianglulated category)

Let $f : A \to B$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. The **mapping cone of** $f$ is the cochain complex $M(f)$ defined by

$$M(f)^n = A^{n+1} \oplus B^n, \qquad d^n_{M(f)} = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}.$$

Define $\alpha(f) : B \to M(f)$ and $\beta(f) : M(f) \to A[1]$ by

$$\alpha(f)^n = \begin{bmatrix} 0 \\ 1_{B^n} \end{bmatrix}, \qquad \beta(f)^n = \begin{bmatrix} 1_{A^{n+1}} & 0 \end{bmatrix}.$$

The sequence

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1]$$

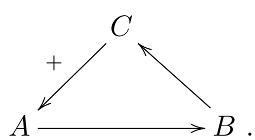
in $\mathbf{K}(\mathcal{A})$ is the **standard triangle** determined by $f$.

The sequences $A \to B \to C \to A[1]$ and $A' \to B' \to C' \to A'[1]$ in $\mathbf{K}(\mathcal{A})$ are **isomorphic** if there exists a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ {\scriptstyle x}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

with vertical arrows isomorphisms in $\mathbf{K}(\mathcal{A})$.

A **triangle** in $\mathbf{K}(\mathcal{A})$ is a sequence $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ that is isomorphic to a standard triangle. We write such a triangle as $(a, b, c)$, $(A, B, C)$, or




The triangles in $\mathbf{K}(\mathcal{A})$ satisfy the following properties:

Theorem.

(T1) *A sequence $A \to B \to C \to A[1]$ that is isomorphic to a triangle is also a triangle.*

(T2) *For each object $A$, the sequence $0 \to A \xrightarrow{1} A \to 0[1]$ is a triangle.*

(T3) *If $(a, b, c)$ is a triangle, then so are $(b, c, -a[1])$ and $(-c[-1], a, b)$.*

(T4) *If $(a, b, c)$ and $(a', b', c')$ are triangles and there are morphisms $x$ and $y$ such that $a'x = ya$, then there exists a morphism $z$ such that the following diagram is commutative:*

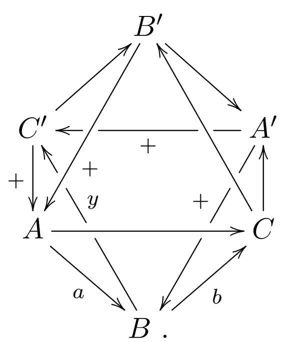
$$\begin{array}{ccccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & A[1] \\ {\scriptstyle x}\downarrow & & {\scriptstyle y}\downarrow & & {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle x[1]} \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & A'[1] \ . \end{array}$$

*(T5) For each pair of morphisms* $A \xrightarrow{a} B \xrightarrow{b} C$*, there is a commutative diagram*

$$
\begin{array}{ccccccc}
A & \xrightarrow{a} & B & \xrightarrow{y} & C' & \longrightarrow & A[1] \\
{\scriptstyle 1}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
A & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & A[1] \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a[1]} \\
 & & A' & \xrightarrow{1} & A' & \xrightarrow{x} & B[1] \\
 & & {\scriptstyle x}\downarrow & & \downarrow & & \\
 & & B[1] & \xrightarrow{y[1]} & C'[1] \, , & &
\end{array}
$$

*where the first two rows and the two middle columns are triangles.*

Property (T5), the "Octahedral axiom," can be visualized using the following diagram:




The four triangles with arrows going in the same direction are triangles; the other ones are commutative. The slanted squares $ABA'B'$ and $BCB'C'$ are commutative.

A **triangulated category** is an additive category $\mathcal{T}$ with an auto-equivalence $\mathcal{T} \to \mathcal{T}$, $A \mapsto A[1]$ (with inverse $A \mapsto A[-1]$) together with a class of sequences $A \to B \to C \to A[1]$ in $\mathcal{T}$ called triangles such that (T1)-(T5) above hold. The theorem shows that $\mathbf{K}(\mathcal{A})$ is a triangulated category with the degree one shift functor and the triangles defined as above.

**6.5**. (Localization)

Let $\mathcal{C}$ be a category and let $S$ be a family of morphisms in $\mathcal{C}$. A **localization** of $\mathcal{C}$ with respect to $S$ is a category $\mathcal{C}_S$ and a functor $Q : \mathcal{C} \to \mathcal{C}_S$ such that

1. $Q(s)$ is an isomorphism for each $s$ in $S$,
2. if $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism for each $s$ in $S$, then there exists a unique functor $F' : \mathcal{C}_S \to \mathcal{D}$ such that $F' \circ Q = F$.

Such a localization is unique up to equivalence. We give conditions on $S$ that guarantee the existence of the pair $(\mathcal{C}_S, Q)$.

The family $S$ is a **multiplicative system** if it satisfies the following properties.

(S1) $1_A \in S$ for each object $A$ of $\mathcal{C}$.

(S2) If $f$ and $g$ are in $S$, then $gf$ is in $S$ if defined.

(S3) Any diagrams

$$\begin{array}{ccc} B & & \\ \uparrow & & \\ A & \Longrightarrow & C \end{array} \qquad\qquad \begin{array}{ccc} B & & \\ \downarrow & & \\ A & \Longleftarrow & C \end{array}$$

with horizontal arrows in $S$ can be completed to commutative diagrams

$$\begin{array}{ccc} B & \Longrightarrow & D \\ \uparrow & & \uparrow \\ A & \Longrightarrow & C \end{array} \qquad\qquad \begin{array}{ccc} B & \Longleftarrow & D \\ \downarrow & & \downarrow \\ A & \Longleftarrow & C \end{array}$$

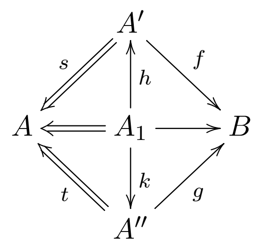
with horizontal arrows in $S$.

(S4) If $f$ and $g$ are morphisms in $\mathcal{C}$, then there exists $s$ in $S$ such that $sf = sg$ if and only if there exists $t$ in $S$ such that $ft = gt$.

Assume that $S$ is a multiplicative system. Let $A$ and $B$ be objects of $\mathcal{C}$. A pair $(s, f)$ of morphisms

$$A \overset{s}{\Longleftarrow} A' \overset{f}{\longrightarrow} B$$

with $s$ in $S$ is a **(right) fraction** from $A$ to $B$. Two fractions $(s, f)$ and $(t, g)$ from $A$ to $B$ are **equivalent**, written $(s, f) \sim (t, g)$, if there exists a commutative diagram

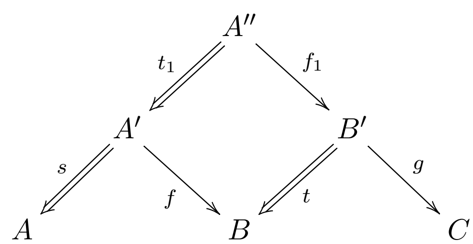


with left horizontal arrow in $S$. (Put more simply, the fractions are equivalent if there exist $h$ and $k$ making the left and right triangles commute and such that $sh = tk$ is in $S$. The horizontal arrows are shown only to indicate that a new fraction arises.) One shows that $\sim$ is an equivalence relation. Denote by $s^{-1}f$ the equivalence class of the fraction $(s, f)$.

If $(s, f)$ is a fraction from $A$ to $B$ and $(t, g)$ is a fraction from $B$ to $C$, put

$$t^{-1}g \circ s^{-1}f = (st_1)^{-1}(gf_1),$$

where $t_1$ and $f_1$ are as in (S3):



One shows that this definition is independent of the choices made.

For objects $A$ and $B$ of $\mathcal{C}$, let $\mathrm{Hom}_{\mathcal{C}_S}(A, B)$ be the class of all $s^{-1}f$ with $(s, f)$ a fraction from $A$ to $B$. If this class is a set for every $A$ and $B$, then, using the composition defined above, we get a category $\mathcal{C}_S$ (having the same objects as $\mathcal{C}$).

Theorem. *If $\mathrm{Hom}_{\mathcal{C}_S}(A, B)$ is a set for each $A$ and $B$, then $\mathcal{C}_S$ together with the functor $Q : \mathcal{C} \to \mathcal{C}_S$ that is the identity on objects and that sends the morphism $f$ to $1^{-1}f$ is a localization of $\mathcal{C}$ with respect to $S$.*

**6.6.** (Derived category)

Let $A$ and $B$ be cochain complexes in the abelian category $\mathcal{A}$. A chain map $f : A \to B$ is a **quasi-isomorphism** if the induced map $H^n(f)$ :

$H^n(A) \to H^n(B)$ is an isomorphism for each $n$. A homotopy class of chain maps is a **quasi-isomorphism** if any (and hence every) representative is a quasi-isomorphism. Let $S$ be the family of all quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

THEOREM. *The family $S$ is a multiplicative system in $\boldsymbol{K}(\mathcal{A})$.*

The proof of this theorem makes use of the triangulated structure on $\mathbf{K}(\mathcal{A})$ as follows. Let $\mathcal{N}$ be the collection of all exact cochain complexes, that is, the collection of all cochain complexes $C$ for which $H^n(C) = 0$ for each $n$.

LEMMA. *The chain map $f : A \to B$ is a quasi-isomorphism if and only if there exists a triangle*

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

*with $C \in \mathcal{N}$.*

*Proof.* Let $f : A \to B$ be a chain map and let

$$A \xrightarrow{f} B \longrightarrow M(f) \longrightarrow A[1]$$

be the associated standard triangle. The sequence $0 \to B \to M(f) \to A[1] \to 0$ is exact and so it gives rise to the long exact sequence

$$\cdots \to H^{n-1}(A[1]) \to H^n(B) \to H^n(M(f)) \to H^n(A[1]) \to H^{n+1}(B) \to \cdots$$

which is the same as

$$\cdots \to H^n(A) \to H^n(B) \to H^n(M(f)) \to H^{n+1}(A) \to H^{n+1}(B) \to \cdots .$$

A straightforward computation shows that the connecting morphisms are induced by $f$. Therefore, if $f$ is a quasi-isomorphism, then $M(f)$ is in $\mathcal{N}$. The proof of the converse is similar. $\square$

The **derived category** of $\mathcal{A}$, denoted $\mathbf{D}(\mathcal{A})$, is the localization $\mathbf{K}(\mathcal{A})_S$ provided this localization is defined, that is, provided the classes $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})_S}(A, B)$ are all sets (which is the case if $\mathcal{A} = R$-**mod** with $R$ a ring).

$\mathbf{D}(\mathcal{A})$ inherits the structure of triangulated category from $\mathbf{K}(\mathcal{A})$.

**6.7**. (Ext)

In this section, we assume that the abelian category $\mathcal{A}$ has enough injectives (or enough projectives with suitable adjustments). In the following theorem, on the right hand side we identify the object $A$ of $\mathcal{A}$ with the complex having degree zero term $A$ and zeros elsewhere, and similarly for $B$.

THEOREM. *For every integer $n$, we have $\mathrm{Ext}^n_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\boldsymbol{D}(\mathcal{A})}(A, B[n])$.*

The proof depends on two lemmas. A (cochain) complex $I$ is **bounded below** if $I^n = 0$ for all $n \ll 0$.

LEMMA. *If $I$ is a bounded below complex of injective objects and $t : I \to X$ is a quasi-isomorphism, then there exists $s : X \to I$ with $st \sim 1_I$.*

*Proof.* Since $t$ is a quasi-isomorphism, it follows that the mapping cone $M(t) = I[1] \oplus X$ (see 6.4) is exact. The map $\beta(t) : M(t) \to I[1]$ is seen to be null homotopic by the proof of the Comparison theorem. If $(u, s)$ from $I[1] \oplus X$ to $I$ is a corresponding homotopy, then $s$ is the desired map and $u$ is the homotopy showing that $st \sim 1_I$. □

LEMMA. *If $I$ is a bounded below complex of injective objects, then the canonical map*

$$\varphi : \mathrm{Hom}_{\boldsymbol{K}(\mathcal{A})}(A, I) \to \mathrm{Hom}_{\boldsymbol{D}(\mathcal{A})}(A, I)$$

*is an isomorphism for every complex $A$.*

*Proof.* If $(t, f)$ is a fraction and $s$ is as in the preceding lemma, then

$$\varphi(sf) = 1^{-1}(sf) = 1^{-1}(st)t^{-1}f = t^{-1}f$$

so $\varphi$ is surjective.

Let $f, g : A \to I$ and assume that $1^{-1}f = \varphi(f) = \varphi(g) = 1^{-1}g$. Then $ft_1 = gt_1$ for some quasi-isomorphism $t_1$, so that $tf = tg$ for some quasi-isomorphism $t$ by (S4) of 6.5. With $s$ as in the preceding lemma, we get $f \sim stf = stg \sim g$, so $\varphi$ is injective. □

The proof of the theorem now goes as follows. The object $B$ has an injective resolution $B \to I$, which we can write

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array} .$$

This chain map is a quasi-isomorphism, which is invertible in $\mathbf{D}(\mathcal{A})$. Therefore,

$$\begin{aligned} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[n]) &\cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, I[n]) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, I[n]) \qquad \text{(by Lemma)} \\ &\cong H^n(\operatorname{Hom}_{\mathcal{A}}(A, I)) \\ &= \operatorname{Ext}^n_{\mathcal{A}}(A, B), \end{aligned}$$

where the third isomorphism sends the homotopy class $\bar{f}$ to the class of $f^0$.