Homological Algebra I

Randall R. Holmes

January 12, 2011

1 Introduction

1.1. A finite group G has a "composition series", that is, a sequence of subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

with each quotient G_i/G_{i+1} simple. These quotients are the "composition factors" of G.

The composition factors of a group do not, in general, determine the group up to isomorphism. For instance, both $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $\mathbf{Z}_4$ have the same composition factors, $\mathbf{Z}_2, \mathbf{Z}_2$, yet they are not isomorphic. One imagines that the simple groups $\mathbf{Z}_2$ and $\mathbf{Z}_2$ are stacked differently to form these two groups.

The classification theorem of finite simple groups gives a list of all the finite simple groups, so the problem that remains for a classification of all finite groups is a way to describe all possible ways to stack a list of finite simple groups to form other groups.

Homological algebra provides a methodology for tackling this "stacking problem". (For simplicity in this introduction, we just address the case of abelian groups.) Associated to two abelian groups A and C is a certain abelian group $\mathrm{Ext}^1_{\mathbf{Z}}(C, A)$, the elements of which parametrize the essentially different ways to stack C above A to form an abelian group. It is shown that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_2, \mathbf{Z}_2) \cong \mathbf{Z}_2 = \{0, 1\}$. The element 0 corresponds to the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the element 1 corresponds to the group $\mathbf{Z}_4$.

2 Chain complex and homology

Throughout, R denotes a ring and R-module means left R-module.

2.1. A sequence of R-modules and homomorphisms

$$C: \quad \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is a **chain complex** if $d_n d_{n+1} = 0$ for each n, or, equivalently, $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for each n. The maps d_n are called **differentials**. The index n in C_n (or d_n) is called the **degree** of the module (or map).

Let C be a chain complex. For each n, put $Z_n = Z_n(C) = \ker d_n$ (elements are called "cycles"), $B_n = B_n(C) = \operatorname{im} d_{n+1}$ (elements are called "boundaries"), and

$$H_n = H_n(C) = Z_n/B_n,$$

the n**th homology module** of C. If $H_n(C) = 0$ for each n, the complex C is **acyclic**.

Exercise. With $R = \mathbf{Z}$, compute the homology modules for

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbf{Z}_8 \xrightarrow{\times 4} \mathbf{Z}_8 \xrightarrow{\times 4} \mathbf{Z}_8 \xrightarrow{\times 4} \cdots$$

(assuming that the first $\mathbf{Z}_8$ is the degree 0 term).

2.2. (Singular Homology)

Homology had its origin in topology as a measure of the hole structure of a topological space.

In R^n, let e_0 denote the origin and let e_i be the point $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$. The **standard** n**-simplex**, denoted $\triangle_n$, is the convex hull of the set $\{e_0, e_1, \ldots, e_n\}$:

$$\triangle_n = [e_0, e_1, e_2, \ldots, e_n] := \left\{ \sum_{i=1}^{n} t_i e_i \;\middle|\; t_i \geq 0, \sum t_i \leq 1 \right\}.$$

For instance, $\triangle_0$ is a point, $\triangle_1$ is the unit interval in R^1, $\triangle_2$ is a triangle (with interior) in R^2, and $\triangle_3$ is a tetrahedron (with interior) in R^3.

Let X be a topological space. An n**-simplex** in X is a continuous map $\sigma : \triangle_n \to X$ (so an n-simplex can be thought of as a distortion of the standard n-simplex $\triangle_n$ in X). Let $S_n(X)$ be the free abelian group on all n-simplexes in X.

We view the standard m-simplex $[e_0, e_1, \ldots, e_m]$ as being "ordered," meaning that we retain the list of vertices in the indicated order. This allows us to regard the (ordered) convex hull $[p_0, p_1, \ldots, p_m]$ of any $m+1$ points in R^n as the m-simplex $\sigma : \triangle_m \to \mathrm{R}^n$ given by

$$\sigma\left(\sum_{i=1}^{m} t_i e_i\right) = \left(1 - \sum_{i=1}^{m} t_i\right) p_0 + \sum_{i=1}^{m} t_i p_i.$$

(This is the natural vertex-order-preserving map.) For instance, with this convention $[p_0, p_1]$ is regarded as a 1-simplex that maps the unit interval to the line segment joining p_0 and p_1 mapping e_0 to p_0 and e_1 to p_1. Note that the standard simplex $\triangle_n = [e_0, e_1, \ldots, e_n]$ is thus viewed as an n-simplex in R^n with map given by the inclusion map.

Consider the standard 2-simplex $\triangle_2 = [e_0, e_1, e_2]$, a triangle. The "boundary" of $\triangle_2$, denoted $d_2(\triangle_2)$, is $[e_1, e_2] - [e_0, e_2] + [e_0, e_1]$, an element of the group $S_1(\mathrm{R}^2)$.

More generally, the n**th boundary operator** $d_n : S_n(X) \to S_{n-1}(X)$ is given by

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ [e_0, \ldots, \hat{e}_i, \ldots, e_n],$$

where $\hat{e}_i$ means "delete e_i."

In general, the composition $d_n d_{n+1}$ is zero. For instance,

$$\begin{aligned} d_1 d_2(\triangle_2) &= d_1([e_1, e_2] - [e_0, e_2] + [e_0, e_1]) \\ &= \big([e_2] - [e_1]\big) - \big([e_2] - [e_0]\big) + \big([e_1] - [e_0]\big) \\ &= 0. \end{aligned}$$

Therefore

$$S(X): \qquad \cdots \longrightarrow S_{n+1}(X) \xrightarrow{d_{n+1}} S_n(X) \xrightarrow{d_n} S_{n-1}(X) \longrightarrow \cdots$$

is a chain complex, called the **singular chain complex of** X.

Take the case $X = \mathrm{R}^2$. The first homology group ($\mathbf{Z}$-module) of this chain complex is $H_1 = Z_1/B_1 = \ker d_1 / \operatorname{im} d_2$. The computation above shows that the boundary of the triangle $\triangle_2$ is in $\ker d_1$. But that boundary is also in $\operatorname{im} d_2$ since it equals $d_2(\triangle_2)$. Therefore the boundary of $\triangle_2$ represents the trivial element of H_1. Now $\ker d_1$ contains not only boundaries of triangles, but other "1-cycles" as well, but they are also in $\operatorname{im} d_2$ (for instance, a polygon is in $\ker d_1$, but it is the boundary of its triangulation since the inner edges cancel). In short, H_1 is the trivial group.

Things change if one removes a point from the plane, say, a point P interior to the standard 2-simplex $\triangle_2$. In this case, there is no 2-simplex in $\mathrm{R}^2 - P$ having boundary equal to the boundary of $\triangle_2$. Therefore H_1 is nontrivial.

Now take the case $X = \mathrm{R}^3 - P$ where P is a point interior to the standard 3-simplex (tetrahedron). Here, H_1 is trivial (now the boundary of a 2-simplex in R^3 is the boundary of some 2-simplex in X since this simplex can distort the interior of the triangle to miss the hole), however, H_2 is nontrivial, since the boundary of the tetrahedron $\triangle_3$ is in $\ker d_2$ but it is not the boundary of a 3-simplex in X.

Exercise. Let X be the punctured plane $\mathrm{R}^2 - \{(0,0)\}$. Convince yourself that the boundaries of any two triangles with the origin in their interiors represent the same element of H_1.

2.3. Let C and C' be chain complexes. A **chain map** f from C to C' (denoted $f : C \to C'$) is a sequence $\{f_n\}$ of R-homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \xrightarrow{d'_{n-1}} & \cdots
\end{array}$$

A chain map $f : C \to C'$ induces well-defined maps $Z_n(f) : Z_n(C) \to Z_n(C')$, $B_n(f) : B_n(C) \to B_n(C')$, and $H_n(f) : H_n(C) \to H_n(C')$ for each n.

Denote by $\mathbf{Ch}(R\text{-}\mathbf{mod})$ the category having chain complexes of R-modules as objects and chain maps as morphisms.

THEOREM. *Z_n, B_n, and H_n are functors from $\mathbf{Ch}(R\text{-}\mathbf{mod})$ to R-$\mathbf{mod}$ for each n.*

2.4. (Chain homotopy)

A chain map $f : C \to C'$ is **null homotopic** if there exist homomorphisms $s_n : C_n \to C'_{n+1}$ such that $f_n = s_{n-1}d_n + d'_{n+1}s_n$ for each n:

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Two chain maps $f, g : C \to C'$ are **homotopic** (written $f \sim g$) if $f - g$ is null homotopic, that is, if there exist homomorphisms $s_n : C_n \to C'_{n+1}$ such that $f - g = sd + d's$ (subscripts suppressed). In this case, the sequence $\{s_n\}$ is called a **chain homotopy** from f to g. The relation $\sim$ is an equivalence relation. A chain map f is null homotopic if and only if $f \sim 0$.

The following theorem says that homotopic chain maps induce the same maps on homology.

Theorem. *Let $f, g : C \to C'$ be chain maps. If $f \sim g$, then $H_n(f) = H_n(g)$ for each n.*

The term "homotopic" comes from topology. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Composing n-simplexes with f produces a homomorphism $S_n(X) \to S_n(Y)$ for each $n \geq 0$ (see 2.2), and these homomorphisms form a chain map $f_* : S(X) \to S(Y)$. If $g : X \to Y$ is another continuous map homotopic to f (meaning there exists a continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, i.e., f can be continuously deformed to g), then f_* and g_* are chain homotopic (and therefore induce the same maps on homology).

A topological space X is "contractible" if the identity map $1 : X \to X$ is homotopic to the map $g : X \to \{x_0\}$ for some $x_0 \in X$. In this case, $g_* = 0$, so that $1_* \sim 0$, that is, 1_* is null homotopic. By analogy, a chain complex C is **contractible** if the identity chain map $1 : C \to C$ is null homotopic. A contractible chain complex is acyclic.

3 Abelian category

3.1. An **Ab-category** is a category $\mathcal{C}$ with an (additive) abelian group structure on each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ such that morphism composition distributes over addition.

An **additive category** is an **Ab**-category $\mathcal{C}$ with an object that is both initial and terminal (called a "zero object" and denoted 0) and with a product $A \times B$ for each pair (A, B) of objects. This last assumption implies the existence of finite products and finite coproducts.

Let $\mathcal{C}$ be an additive category and let $f : B \to C$ be a morphism.

- A **kernel** of f is a morphism $i : A \to B$ such that $fi = 0$ and such that if $i' : A' \to B$ also satisfies $fi' = 0$, then there exists a unique $j : A' \to A$ for which $i' = ij$.
- A **cokernel** of f is a morphism $p : C \to D$ such that $pf = 0$ and such that if $p' : C \to D'$ also satisfies $p'f = 0$, then there exists a unique $q : D \to D'$ for which $p' = qp$.
- An **image** of f is a kernel of a cokernel of f.
- A **coimage** of f is a cokernel of a kernel of f.

A kernel (if it exists) is unique up to an isomorphism compatible with the structure maps (i.e., if $i : A \to B$ and $i' : A' \to B$ are kernels of $f : B \to C$, then there exists an isomorphism $j : A \to A'$ such that $i'j = i$). A similar statement holds for cokernel, image, and coimage.

We often say *the* kernel of f (denoted $\ker f$) to refer to the equivalence class of all kernels of f or (by abuse of terminology) to any kernel of f. A similar statement holds for $\mathrm{coker}\, f$, $\mathrm{im}\, f$, and $\mathrm{coim}\, f$.

Assume for the moment that $\mathcal{C}$ is a module category. If $\ker f \subseteq B$ has its usual meaning, then the inclusion map $\ker f \to B$ is a kernel of f. If $\mathrm{im}\, f \subseteq C$ has its usual meaning, then the inclusion map $\mathrm{im}\, f \to C$ is an image of f. The canonical epimorphism $C \to C/\mathrm{im}\, f$ is a cokernel of f and the canonical epimorphism $B \to B/\ker f$ is a coimage of f.

- A morphism $i : A \to B$ is **monic** if for each morphism $f : A' \to A$, $if = 0$ implies $f = 0$.

- A morphism $p : C \to D$ is **epic** if for each morphism $g : D \to D'$, $gp = 0$ implies $g = 0$.

Exercise. Prove that every kernel is monic and every cokernel is epic.

3.2. An **abelian category** is an additive category $\mathcal{A}$ for which

(a) each morphism in $\mathcal{A}$ has both a kernel and a cokernel,

(b) each monic morphism in $\mathcal{A}$ is a kernel of its cokernel,

(c) each epic morphism in $\mathcal{A}$ is a cokernel of its kernel.

Example. R-**mod** is an abelian category.

Let $\mathcal{A}$ be an abelian category. Define chain complex and chain map just as before except with $\mathcal{A}$ in place of R-**mod** to get the category $\mathbf{Ch}(\mathcal{A})$.

THEOREM. *$\mathbf{Ch}(\mathcal{A})$ is an abelian category.*

3.3. (Embedding theorem)

An abelian category need not be concrete (i.e., the objects in an abelian category need not have underlying sets). Therefore, establishing results in an arbitrary abelian category can be difficult since one cannot use arguments that make use of elements. The Embedding theorem can sometimes be used to get around this difficulty. It says, in effect, that if the object class of an abelian category is not too big, then one can think of the category as a module category. (See the next subsection for an application.)

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is **faithful** (resp. **full**) if the maps that it induces on the Hom sets are all injective (resp. surjective). If F is both faithful and full, it is **fully faithful**.

The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is **exact** at B if $\ker g = \operatorname{im} f$. A sequence $\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$ is **exact** if it is exact at each object A_n.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. F is **exact** if it preserves exactness, that is, if $A \xrightarrow{f} B \xrightarrow{g} C$ exact implies that $F(A) \xrightarrow{F(f)}$

$F(B) \overset{F(g)}{\to} F(C)$ is exact. Equivalently, F is exact if $0 \to A \to B \to C \to 0$ exact implies $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact. One can show that if F is exact, then it is additive, meaning that its maps on Hom groups are homomorphisms.

Let $\mathcal{C}$ be a category. A collection of objects and morphisms in $\mathcal{C}$ is a **subcategory** if it is closed under composition of morphisms and contains 1_A for each object A in the collection. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is **full** if $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any pair of objects (A, B) in $\mathcal{D}$.

A category is **small** if its class of objects is a set.

THEOREM. *(Freyd-Mitchell Embedding Theorem) If $\mathcal{A}$ is a small abelian category, then there exists a ring R and an exact, fully faithful functor from $\mathcal{A}$ to R-**mod**.*

3.4. The proof of the following lemma uses a common technique in homological algebra called "diagram chasing." The embedding theorem of the previous subsection is used to reduce to the case of a module category.

LEMMA. *(5-Lemma) Let*

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "\alpha"'] & B \arrow[r, "f"] \arrow[d, "\beta"'] & C \arrow[r, "g"] \arrow[d, "\gamma"'] & D \arrow[r, "h"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r, "e'"] & B' \arrow[r, "f'"] & C' \arrow[r, "g'"] & D' \arrow[r, "h'"] & E'
\end{tikzcd}$$

be a commutative diagram in an abelian category $\mathcal{A}$. The following hold:

(i) *if β and δ are monic and α is epic, then γ is monic,*

(ii) *if β and δ are epic and ϵ is monic, then γ is epic,*

(iii) *if α, β, δ, and ϵ are equivalences, then γ is an equivalence.*

Proof. We show only how to reduce to the case where $\mathcal{A}$ is an abelian category. Let $\mathcal{A}'$ be the smallest abelian subcategory of $\mathcal{A}$ containing the objects and morphisms of the diagram. Then $\mathcal{A}'$ is a small category, so there is an exact, fully faithful functor from $\mathcal{A}'$ onto a full subcategory of R-**mod** for some ring R. Therefore, if the lemma is established for $\mathcal{A} = R$-**mod** it holds in general. $\square$

4 Long exact sequence

4.1.

LEMMA. *(Snake lemma) Let*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r, "p"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C & \end{tikzcd}$$

be a commutative diagram with exact rows in an abelian category. There exists an exact sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h\ .$$

If $A' \to B'$ is monic, then so is $\ker f \to \ker g$, and if $B \to C$ is epic, then so is $\operatorname{coker} g \to \operatorname{coker} h$. If the category is a module category, then ∂ is given by $\partial(c') = i^{-1}gp^{-1}(c') + \operatorname{im} f$.

LEMMA. *(3×3 lemma) Let*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

be a commutative diagram with exact columns in an abelian category.

(i) *If the top two rows are exact, then so is bottom.*

(ii) *If the bottom two rows are exact, then so is the top.*

Exercise. Prove the 3×3 lemma.

4.2.

THEOREM. *(Long exact sequence) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of chain complexes in an abelian category $\mathcal{A}$. For each n, there exists a natural morphism $\partial = \partial_n : H_n(C) \to H_{n-1}(A)$ such that*

$$\cdots \xrightarrow{g_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} \cdots$$

is exact (where $f_* = H_n(f)$ etc.).

The morphisms ∂_n are called **connecting morphisms**.

The term "natural morphism" is frequently used as an informal way of indicating a natural transformation between functors. (If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, a **natural transformation** τ from F to G is a collection of morphisms $\tau_A : F(A) \to G(A)$, one for each object A of $\mathcal{C}$, such that, for every morphism $f : A \to B$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau_B"] & G(B) \end{tikzcd}$$

is commutative.) Usually, the intended functors can easily be inferred from the context, but this is not the case in the statement of the last theorem, so we explain: Let $\mathcal{C}$ be the category of exact sequences

$$S : \quad 0 \to A \to B \to C \to 0$$

of chain complexes in $\mathcal{A}$ (the morphisms in $\mathcal{C}$ are the chain maps). For each n, let $F_n, G_n : \mathcal{C} \to \mathcal{A}$ be the functors that send S to $H_n(C)$ and $H_n(A)$, respectively (and a chain map to the induced morphisms). The meaning of "natural morphism" in the statement of the theorem is that ∂_n is a natural transformation from F_n to G_{n-1}. In particular, if

$$\begin{tikzcd} S : \arrow[d] & 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ S' : & 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is a morphism in $\mathcal{C}$, then the diagram

$$\begin{tikzcd} H_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"'] & H_{n-1}(A) \arrow[d, "\alpha_*"'] \\ H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \end{tikzcd}$$

is commutative.

A consequence of the naturality of the connecting morphisms is that the mapping of a short exact sequence to its corresponding long exact sequence (as in the theorem) defines a functor from $\mathcal{C}$ to the category $\mathcal{D}$ of long exact sequences in $\mathcal{A}$. This functor sends the morphism $S \to S'$ (shown above) to the commutative "ladder"

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(C) \arrow[r, "\partial"] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1}(C') \arrow[r, "\partial"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

which is a morphism in $\mathcal{D}$.

5 Projective resolution

Throughout $\mathcal{A}$ denotes an abelian category.

5.1. (Projective object)

An object P of $\mathcal{A}$ is **projective** if given any diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

with exact bottom row, there exists a morphism $P \to B$ (not necessarily unique) making the diagram commutative.

Theorem. *Let P be an object of $\mathcal{A}$. The following are equivalent:*

(i) *P is projective,*

(ii) $\mathrm{Hom}_{\mathcal{A}}(P, \cdot\,)$ *is exact,*

(iii) *every exact sequence $0 \to A \to B \to P \to 0$ in $\mathcal{A}$ splits.*

THEOREM. *Every free R-module is projective.*

The converse of this theorem is not true. We need a theorem in order to give a counterexample.

THEOREM. *An R-module P is projective if and only if $F = P \oplus P'$ for some R-modules F and P' with F free.*

A projective module need not be free ($\mathbf{Z}_6 = \mathbf{Z}_2 \oplus \mathbf{Z}_3$ so $\mathbf{Z}_2$ is a projective $\mathbf{Z}_6$-module that is not free). However, if R is a principal ideal domain, then every projective R-module is free. Also, if R is a division ring, then every R-module is free and hence projective.

Exercise. Prove that the category of *finite* abelian groups has *no* nontrivial projective objects at all.

5.2. A **left resolution** of an object A of $\mathcal{A}$ is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f} A \longrightarrow 0\,. \tag{*}$$

This resolution is often abbreviated $P \xrightarrow{f} A$, where P denotes the complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0\,.$$

(The context clarifies whether P denotes a complex as here or a single object.) The complex P is sometimes called the "deleted" complex of the complex (*). It is exact in each degree with the possible exception of degree 0.

A left resolution as in (*) in which each P_i is projective is a **projective resolution** of A.

We say that $\mathcal{A}$ has **enough projectives** if for each object A there is an epic morphism $P \to A$ with P projective.

Theorem. *If $\mathcal{A}$ has enough projectives, then each object of $\mathcal{A}$ has a projective resolution.*

Theorem. *R-**mod** has enough projectives.*

In particular, each R-module has a projective resolution.

5.3. (Comparison theorem)

Let A and B be two objects of $\mathcal{A}$, let $f : A \to B$ be a morphism, and let $P \to A$ and $Q \to B$ be projective resolutions. A chain map $\bar{f} : P \to Q$ **lifts** f if the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\bar{f}_2"'] & P_1 \arrow[r] \arrow[d, "\bar{f}_1"'] & P_0 \arrow[r] \arrow[d, "\bar{f}_0"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

(it is enough to say that the last square is commutative since the other squares are commutative by the definition of a chain map).

Theorem. *(Comparison theorem) If $f : A \to B$ is a morphism and $P \to A$ and $Q \to B$ are projective resolutions, then there exists a chain map $\bar{f} : P \to Q$ that lifts f, and any two such chain maps are homotopic.*

5.4. (Horseshoe Lemma)

Let

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P'_1 \arrow[d] & & P''_1 \arrow[d] & \\
 & P'_0 \arrow[d, "\epsilon'"'] & & P''_0 \arrow[d, "\epsilon''"'] & \\
0 \arrow[r] & A' \arrow[r, "\iota"] \arrow[d] & A \arrow[r, "\pi"] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

be a diagram in $\mathcal{A}$ where the bottom row is exact and the columns are projective resolutions of A' and A'', respectively. The next theorem says that one can fill in the horseshoe with a projective resolution of A and chain maps in such a way as to create an exact sequence of chain complexes.

THEOREM. *(Horseshoe lemma) There exists a projective resolution $P \xrightarrow{\epsilon} A$ and chain maps $\bar{\iota}$ and $\bar{\pi}$ lifting ι and π, respectively, such that*

$$0 \longrightarrow P' \xrightarrow{\bar{\iota}} P \xrightarrow{\bar{\pi}} P'' \longrightarrow 0$$

is exact. In fact, one can let $P_n = P'_n \oplus P''_n$ and let $\bar{\iota}_n$ and $\bar{\pi}_n$ be the natural injection and projection, respectively.

6 Left derived functor

Throughout, $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor. We assume that $\mathcal{A}$ has enough projectives.

6.1. For each object A of $\mathcal{A}$, choose a projective resolution $P_A \to A$ of A.

The n**th left derived functor of** F is the functor $L_nF : \mathcal{A} \to \mathcal{B}$ defined as follows:

- For $A \in \operatorname{ob} \mathcal{A}$, $\quad L_nF(A) = H_n(F(P_A))$,
- For $f : A \to B$, $\quad L_nF(f) = H_n(F(\bar{f}))$, where $\bar{f} : P_A \to P_B$ lifts f.

L_nF is an additive functor. Note that $L_nF(A) = 0$ for all $n < 0$ (since $(P_A)_n = 0$ for all $n < 0$).

For each object A of $\mathcal{A}$, let $\hat{P}_A \to A$ be a projective resolution of A and let $\hat{L}_nF$ denote the functor with $\hat{P}_A$ replacing P_A in the above definition.

THEOREM. *$\hat{L}_nF$ is naturally equivalent to L_nF for each n.*

COROLLARY. *If P is a projective object, then $L_nF(P) = 0$ for all $n \neq 0$.*

6.2. (Long exact sequence)

THEOREM. *If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{A}$, then there exists an exact sequence*

$$\cdots \xrightarrow{\partial} L_1F(A') \to L_1F(A) \to L_1F(A'') \xrightarrow{\partial} L_0F(A') \to L_0F(A) \to L_0F(A'') \to 0,$$

where the morphisms ∂ are natural.

6.3. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say that F is **right exact** if

$$A \to B \to C \to 0 \text{ exact} \quad \Rightarrow \quad F(A) \to F(B) \to F(C) \to 0 \text{ exact}.$$

For example, if A is a right R-module, then $A \otimes_R \cdot : R$-**mod**$\to$ **Ab** is right exact. However, this functor need not be exact as can be seen by applying $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \cdot$ to $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$.

THEOREM. *L_0F and F are naturally isomorphic if and only if F is right exact.*

6.4. (Example: Tor)

Let A be a right R-module and put $F = A \otimes_R \cdot : R$-**mod**$\to$ **Ab**. Define

$$\mathrm{Tor}_n^R(A, \cdot) = L_nF.$$

Example. For any abelian group A and positive integer m, we have

$$\mathrm{Tor}_n^{\mathbf{Z}}(A, \mathbf{Z}_m) \cong \begin{cases} A/mA, & n = 0, \\ A[m], & n = 1, \\ 0, & n \geq 2, \end{cases}$$

where $A[m] := \{a \in A \,|\, ma = 0\}$.

THEOREM. *If $0 \to B' \to B \to B'' \to 0$ is an exact sequence in R-**mod**, then there exists an exact sequence*

$$\cdots \to \mathrm{Tor}_2^R(A, B'') \to \mathrm{Tor}_1^R(A, B') \to \mathrm{Tor}_1^R(A, B) \to \mathrm{Tor}_1^R(A, B'') \to$$
$$\to A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0$$

7 Injective resolution

Throughout $\mathcal{A}$ denotes an abelian category.

7.1. (Injective object)

An object E of $\mathcal{A}$ is **injective** if given any diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ & E & \end{tikzcd}$$

with exact top row, there exists a morphism $B \to E$ (not necessarily unique) making the diagram commutative.

THEOREM. *Let E be an object of $\mathcal{A}$. The following are equivalent:*

(i) *E is injective,*

(ii) $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, E)$ *is exact,*

(iii) *every exact sequence $0 \to E \to B \to C \to 0$ in $\mathcal{A}$ splits.*

THEOREM. (Baer's Criterion.) *An R-module M is injective if and only if for every left ideal I of R, every R-homomorphism $I \to M$ extends to an R-homomorphism $R \to M$:*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[dl, dashed, "\exists"] \\ & M & \end{tikzcd}$$

7.2. (Divisible module)

An R-module M is **divisible** if for each $m \in M$ and non-(zero divisor) $r \in R$, there exists $n \in M$ such that $rn = m$ (i.e., m can be divided by r). For example, $\mathbf{Q}$ is a divisible $\mathbf{Z}$-module. The class of divisible R-modules is closed under taking direct sums and quotients.

THEOREM.

(i) *Every injective R-module is divisible.*

(ii) *If R is a PID, then every divisible module is injective.*

In particular, $\mathbf{Q}$ is an injective $\mathbf{Z}$-module.

7.3. (Adjoint pair)

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. The pair (F, G) is called an **adjoint pair** if for each $A \in \operatorname{ob}\mathcal{C}$ and $B \in \operatorname{ob}\mathcal{D}$ there exists a bijection

$$\tau_{AB} : \operatorname{Hom}_{\mathcal{D}}(F(A), B) \to \operatorname{Hom}_{\mathcal{C}}(A, G(B))$$

such that $\tau_{A\cdot}$ and $\tau_{\cdot B}$ are both natural transformations.

It can be shown that if $\mathcal{C}$ and $\mathcal{D}$ are additive categories, and (F, G) is an adjoint pair, then the bijections τ_{AB} are automatically group isomorphisms.

The functor F **preserves projectives** if $A \in \operatorname{ob}\mathcal{C}$ projective implies $F(A)$ is projective. The term **preserves injectives** is defined similarly.

THEOREM. *Assume that the categories $\mathcal{C}$ and $\mathcal{D}$ are abelian, the functors F and G are additive, and (F, G) is an adjoint pair.*

(i) *If G is exact, then F preserves projectives.*

(ii) *If F is exact, then G preserves injectives.*

Let R and S be rings and let A_R, ${}_RB_S$, and C_S be modules as indicated. Then $A \otimes_R B$ is a right S module with action given by $(a \otimes b)s = a \otimes (bs)$ and $\operatorname{Hom}_S(B, C)$ is a right R-module with action given by $(fr)(b) = f(rb)$.

THEOREM. *Let*

$$F = \,\cdot\, \otimes_R B : \textbf{mod-}R \to \textbf{mod-}S,$$
$$G = \operatorname{Hom}_S(B, \,\cdot\,) : \textbf{mod-}S \to \textbf{mod-}R.$$

Then (F, G) is an adjoint pair. In particular, for each pair of modules (A_R, C_S) there is a group isomorphism

$$\operatorname{Hom}_S(A \otimes_R B, C) \cong \operatorname{Hom}_R(B, \operatorname{Hom}_S(B, C)).$$

Let $S = \mathbf{Z}$ and $B = {}_R R_Z$ and note that $\cdot \otimes_R R : \mathbf{mod}\text{-}R \to \mathbf{Ab}$ is the forgetful functor. By the previous theorem, $\mathrm{Hom}_{\mathbf{Z}}(R, E)$ is an injective R-module for every injective $\mathbf{Z}$-module E.

7.4. A **right resolution** of an object A of $\mathcal{A}$ is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots . \qquad (*)$$

(By convention, E^i means E_{-i}, so it continues to be the case that indices decrease to the right.) This resolution is often abbreviated $A \xrightarrow{f} E$, where E denotes the complex

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots .$$

A right resolution as in (*) in which each E^i is injective is an **injective resolution** of A.

We say that $\mathcal{A}$ has **enough injectives** if for each object A there is a monic morphism $A \to E$ with E injective. If $\mathcal{A}$ has enough injectives, then each object of $\mathcal{A}$ has an injective resolution.

THEOREM. *R-**mod** has enough injectives.*

(The proof uses 7.2 and 7.3.)

In particular, each R-module has an injective resolution.

8 Right derived functor

8.1. (Opposite category)

Let $\mathcal{C}$ be a category. The **opposite category** of $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is the category having the same object class and morphism class as $\mathcal{C}$, but with the source and target maps reversed. Put another way, $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with the arrows turned around. Given a morphism $f : A \to B$ in $\mathcal{C}$, it is convenient to denote the same morphism when viewed as a morphism in $\mathcal{C}^{\mathrm{op}}$ by $f' : B \to A$, so that we can write, for instance, $(fg)' = g'f'$.

A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ can be regarded as a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ by putting $F(f') = F(f)$.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is the functor that is identical to F on objects and morphisms (so it sends a reversed arrow in $\mathcal{C}$ to the corresponding reversed arrow in $\mathcal{D}$).

Theorem. *If $\mathcal{A}$ is an abelian category and f is a morphism in $\mathcal{A}$, then*

(i) $(\ker f)' = \operatorname{coker} f'$,

(ii) $(\operatorname{coker} f)' = \ker f'$.

An object of $\mathcal{C}$ is projective (resp., injective) if and only if it is injective (resp., projective) when viewed in $\mathcal{C}^{\mathrm{op}}$.

8.2.

Theorem. *(Comparison theorem) If $f : A \to B$ is a morphism and $A \to E$ and $B \to F$ are injective resolutions, then there exists a chain map $\bar{f} : E \to F$ that lifts f, and any two such chain maps are homotopic.*

For the remainder of the section, $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor. We assume that $\mathcal{A}$ has enough injectives. For each object A of $\mathcal{A}$, choose an injective resolution $E_A \to A$ of A.

The **nth right derived functor of** F is the functor $R^nF : \mathcal{A} \to \mathcal{B}$ defined as follows:

- For $A \in \operatorname{ob}\mathcal{A}$, $\quad R^nF(A) = H^n(F(E_A))$,
- For $f : A \to B$, $\quad R^nF(f) = H^n(F(\bar{f}))$, where $\bar{f} : E_A \to E_B$ lifts f.

R^nF is an additive functor. Note that $R^nF(A) = 0$ for all $n < 0$ (since $(E_A)^n = 0$ for all $n < 0$).

Theorem. *R^nF is naturally equivalent to $(L_nF^{\mathrm{op}})^{\mathrm{op}}$ for each n.*

In particular, if $\hat{R}^nF$ is the functor defined using a different choice of injective resolutions, then $\hat{R}^nF$ is naturally equivalent to R^nF.

Corollary. *If E is an injective object, then $R^nF(E) = 0$ for all $n \neq 0$.*

8.3. (Long exact sequence)

Theorem. *If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{A}$, then there exists an exact sequence*

$$0 \to R^0F(A') \to R^0F(A) \to R^0(A'') \xrightarrow{\partial} R^1F(A') \to R^1F(A) \to R^1F(A'') \to \cdots,$$

where the morphisms ∂ are natural.

8.4. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say that F is **left exact** if

$$0 \to A \to B \to C \text{ exact} \quad \Rightarrow \quad 0 \to F(A) \to F(B) \to F(C) \text{ exact.}$$

For example, if M is an R-module, then $\mathrm{Hom}_R(M, \cdot\,) : R\text{-}\mathbf{mod} \to \mathbf{Ab}$ is left exact. However, this functor need not be exact as can be seen by applying $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_2, \cdot\,)$ to $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$.

Theorem. *R^0F and F are naturally isomorphic if and only if F is left exact.*

8.5. (Example: Ext)

Let A be an R-module and put $F = \mathrm{Hom}_R(A, \cdot\,) : R\text{-}\mathbf{mod} \to \mathbf{Ab}$. Define

$$\mathrm{Ext}^n_R(A, \cdot\,) = R^nF : R\text{-}\mathbf{mod} \to \mathbf{Ab}$$

Theorem. *If $0 \to B' \to B \to B'' \to 0$ is an exact sequence in R-**mod**, then there exists an exact sequence*

$$\begin{aligned} &0 \to \mathrm{Hom}_R(A, B') \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B'') \to \\ &\quad \to \mathrm{Ext}^1_R(A, B') \to \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A, B'') \to \mathrm{Ext}^2_R(A, B') \to \cdots \end{aligned}$$

8.6. (Contravariance)

Let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant additive functor and assume that $\mathcal{A}$ has enough projectives. We can regard F as a covariant functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Since $\mathcal{A}^{\mathrm{op}}$ has enough injectives, the functor $R^nF : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is defined (and it is contravariant if the domain is taken to be $\mathcal{A}$).

THEOREM. *If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{A}$, then there exists an exact sequence*

$$0 \to R^0F(A'') \to R^0F(A) \to R^0(A') \xrightarrow{\partial} R^1F(A'') \to R^1F(A) \to R^1F(A') \to \cdots,$$

where the morphisms ∂ are natural.

The contravariant functor F is **left exact** if

$$A \to B \to C \to 0 \text{ exact} \quad \Rightarrow \quad 0 \to F(C) \to F(B) \to F(A) \text{ exact.}$$

For example, if M is an R-module, then $\mathrm{Hom}_R(\,\cdot\,, M) : R\text{-}\mathbf{mod} \to \mathbf{Ab}$ is left exact.

THEOREM. *R^0F and F are naturally isomorphic if and only if F is left exact.*

8.7. (Contravariant Ext)

Let B be an R-module and put $F = \mathrm{Hom}_R(\,\cdot\,, B) : R\text{-}\mathbf{mod} \to \mathbf{Ab}$. Define

$$\mathrm{Ext}^n_R(\,\cdot\,, B) = R^nF : R\text{-}\mathbf{mod} \to \mathbf{Ab}$$

THEOREM. *If $0 \to A' \to A \to A'' \to 0$ is an exact sequence in R-**mod**, then there exists an exact sequence*

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(A'', B) \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A', B) \to \\ &\to \mathrm{Ext}^1_R(A'', B) \to \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A', B) \to \mathrm{Ext}^2_R(A'', B) \to \cdots \end{aligned}$$

We have two definitions for the notation $\mathrm{Ext}^n_R(A, B)$:

$$\mathrm{Ext}^n_R(A, \,\cdot\,)(B) \quad \text{and} \quad \mathrm{Ext}^n_R(\,\cdot\,, B)(A).$$

It turns out that these groups are isomorphic, so there is no ambiguity. (There is an easy proof using the notion of a "spectral sequence.")

Example. For any abelian group A and positive integer m, we have

$$\mathrm{Ext}^n_{\mathbf{Z}}(\mathbf{Z}_m, A) \cong \begin{cases} A[m], & n = 0, \\ A/mA, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

where $A[m] := \{a \in A \mid ma = 0\}$.

9 Direct limit

9.1. Let $(I, \leq)$ be a preordered set ($\leq$ is reflexive and transitive). I gives rise to a category, also denoted I, having the elements of I as objects and with

$$\mathrm{Hom}_I(i, j) = \begin{cases} \{p_i^j\}, & i \leq j \\ \emptyset, & \text{otherwise.} \end{cases}$$

A **direct system** in the category $\mathcal{C}$ with index set I is a functor $A : I \to \mathcal{C}$.

Let $A : I \to \mathcal{C}$ be a direct system. We write A_i for $A(i)$ and α_i^j for $A(p_i^j)$. A **co-cone** of A is a pair $(C, \{g_i\})$ where $C \in \mathrm{ob}\,\mathcal{C}$ and the morphisms $g_i : A_i \to C$ satisfy $g_j \alpha_i^j = g_i$ whenever $i \leq j$.

A **direct limit** of the direct system A is a co-cone $(\varinjlim A_i, \{f_i\})$ of A such that for every co-cone $(C, \{g_i\})$ of A there exists a unique morphism $\varphi : \varinjlim A_i \to C$ such that $\varphi f_i = g_i$ for all $i \in I$:

$$\begin{tikzcd}
\varinjlim A_i \arrow[rr, dashed, "\exists!\,\varphi"] & & C \\
& A_j \arrow[ul, "f_j"'] \arrow[ur, "g_j"] & \\
& A_i \arrow[u, "\alpha_i^j"] \arrow[uul, "f_i"] \arrow[uur, "g_i"'] &
\end{tikzcd}$$

(A direct limit is an example of a "colimit," which is defined the same way, but with I being an arbitrary category.)

THEOREM. *For any direct system in R-**mod**, a direct limit exists.*

9.2. (Examples)

- If the preorder $\leq$ is trivial, then $\varinjlim A_i = \coprod_{i\in I} A_i$.
- If $I = \{1,2,3\}$ with order $1 \leq 2, 1 \leq 3$, then $\varinjlim A_i$ is called the **pushout** of the morphisms α_1^2 and α_1^3.
- If $f : A \to B$ is a morphism, then $\operatorname{coker} f$ is a pushout of f and $0 : A \to 0$ and hence a direct limit.
- If I is the set of subgroups $\langle 1/q\rangle$ of $\mathbf{Q}$ with $0 \neq q \in \mathbf{Z}$ ordered by inclusion, then $\varinjlim A_i \cong \mathbf{Q}$.

9.3. (Direct limit is right exact)

Let I be a preordered set and let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^I$ the category with all direct systems of I in $\mathcal{C}$ as objects and with natural transformations as morphisms. If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^I$.

Assume that $\mathcal{C}$ has a direct limit for each direct system of I in $\mathcal{C}$. Then we get a functor $\varinjlim : \mathcal{C}^I \to \mathcal{C}$ that sends each direct system to its direct limit and sends a morphism of direct systems to the induced morphism. Define the **diagonal functor** $\triangle : \mathcal{C} \to \mathcal{C}^I$ by $(\triangle M)_i = M$ for all i and $\alpha_i^j = 1_M$ for $i \leq j$.

THEOREM. $(\varinjlim, \triangle)$ *is an adjoint pair.*

In particular, if $\mathcal{C}$ is abelian, then $\varinjlim$ is right exact.

9.4. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and assume that direct limits exist in $\mathcal{C}$ and $\mathcal{C}'$. For any preordered set I, the functor F induces a functor $F : \mathcal{C}^I \to (\mathcal{C}')^I$ with $F(A)_i = F(A_i)$ and $F(\tau)_i = F(\tau_i)$ for each direct system A and morphism $\tau : A \to B$. We say that F **preserves direct limits** if for each preordered set I, $\varinjlim F \cong F \varinjlim$ as functors from $\mathcal{C}^I$ to $\mathcal{C}'$.

THEOREM. *If (F, G) is an adjoint pair for some $G : \mathcal{C}' \to \mathcal{C}$, then F preserves direct limits.*

In particular, if F is a left adjoint and $\mathcal{C}$ and $\mathcal{C}'$ are abelian, then F is right exact (since a cokernel is a direct limit).

THEOREM. *The following are equivalent for a functor $F : \mathbf{mod}\text{-}R \to \mathbf{Ab}$:*

(i) *F preserves direct limits,*

(ii) *F is right exact and preserves coproducts,*

(iii) *$F \cong \cdot \otimes_R B$ for some R-module B,*

(iv) *(F, G) is an adjoint pair for some $G : \mathbf{Ab} \to \mathbf{mod}\text{-}R$*

The implication (ii)$\Rightarrow$(iii) is known as Watts' theorem.

9.5. (Directed set)

A preordered set $(I, \leq)$ is **directed** if for each $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

LEMMA. *Let $A : I \to \mathbf{mod}\text{-}R$ be a direct system with $(I, \leq)$ a directed set and let $(\varinjlim A_i, f_i)$ be a direct limit of A.*

(i) *If $a \in \varinjlim A_i$, then $a = f_i(a_i)$ for some $i \in I$ and $a_i \in A_i$,*

(ii) $\ker f_i = \bigcup_{j \geq i} \ker \alpha_i^j$

THEOREM. *If $(I, \leq)$ is directed and $\mathcal{A}$ is an abelian category, then $\varinjlim : \mathcal{A}^I \to \mathcal{A}$ is exact.*

10 Tor

10.1. (Balance)

For a right R-module A, the functor $\mathrm{Tor}_n^R(A, \cdot\,) : R\text{-}\mathbf{mod} \to \mathbf{Ab}$ was defined in 6.4 by

$$\mathrm{Tor}_n^R(A, \cdot\,) = L_n(A \otimes_R \cdot\,).$$

Similarly, for an R-module B, define $\mathrm{Tor}_n^R(\cdot\,, B) : \mathbf{mod}\text{-}R \to \mathbf{Ab}$ by

$$\mathrm{Tor}_n^R(\cdot\,, B) = L_n(\cdot \otimes_R B).$$

These definitions give two meanings to the notation $\operatorname{Tor}_n^R(A, B)$:

$$\operatorname{Tor}_n^R(A, \cdot\,)(B) \quad \text{and} \quad \operatorname{Tor}_n^R(\,\cdot\,, B)(A).$$

It turns out that these groups are isomorphic, so there is no ambiguity. (There is an easy proof using the notion of a "spectral sequence.")

10.2. We translate some elementary properties of left derived functors in general to the Tor notation:

(i) $\operatorname{Tor}_n^R(A, B) = 0$ if $n < 0$,

(ii) if A is projective, then $\operatorname{Tor}_n^R(A, B) = 0$ for all $n \neq 0$,

(iii) $\operatorname{Tor}_0^R(A, \cdot\,) \cong A \otimes_R \cdot$ (since tensor is right exact),

(iv) if $0 \to B' \to B \to B'' \to 0$ is exact, then for each A there is a long exact sequence

$$\cdots \to \operatorname{Tor}_1^R(A, B'') \to A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0.$$

These statements have counterparts in the other variable. To see this, one could use either the preceding section or the following theorem:

Theorem. $\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{R^{\mathrm{op}}}(B, A)$.

10.3.

Theorem. *Let B be an R-module.*

(i) *If $\{A_i\}_{i\in I}$ is a family of right R-modules, then*

$$\operatorname{Tor}_n^R\left(\coprod A_i, B\right) \cong \coprod \operatorname{Tor}_n^R(A_i, B).$$

(ii) *If $A : I \to$ **mod**-R is a direct system and I is directed, then*

$$\operatorname{Tor}_n^R(\varinjlim A_i, B) \cong \varinjlim \operatorname{Tor}_n^R(A_i, B).$$

10.4. (Flat module)

A right R-module Q is **flat** if the functor $Q \otimes_R \cdot$ is exact. A flat (left) R-module is defined similarly. The R-module R_R is flat.

THEOREM. *If $\{M_i\}_{i\in I}$ is a family of R-modules, then $\coprod_i M_i$ is flat if and only if each M_i is flat.*

In particular, every projective module is flat.

THEOREM. *Let $(I,\leq)$ be a directed set and let $A : I \to R$-**mod** be a direct system. If A_i is flat for each i, then $\varinjlim A_i$ is flat.*

The $\mathbf{Z}$-module $\mathbf{Q}$ is flat (it is a direct limit of copies of $\mathbf{Z}$) but it is not projective.

THEOREM. *Let B be an R-module. The following are equivalent:*

(i) *B is flat,*

(ii) $\operatorname{Tor}_n^R(A,B) = 0$ *for all A and all $n \neq 0$,*

(iii) $\operatorname{Tor}_1^R(A,B) = 0$ *for all A.*

10.5. (Acyclicity)

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume that $\mathcal{A}$ has enough projectives. An object Q of $\mathcal{A}$ is (left) F**-acyclic** if $L_nF(Q)$ for all $n \neq 0$. The next theorem says that the left derived functors of F can be computed using F-acyclic resolutions (not just projective resolutions).

THEOREM. *If $Q \to A$ is a left F-acyclic resolution of $A \in \operatorname{ob}\mathcal{A}$, then $L_nF(A) \cong H_n(F(Q))$.*

By 10.4, a flat R-module is acyclic for the functor $A \otimes_R \cdot$ for any A.

COROLLARY. *Let A_R and ${}_RB$ be R-modules as indicated. If $Q \to B$ is a left flat resolution of the R-module B, then $\operatorname{Tor}_n^R(A,B) = H_n(A \otimes_R Q)$.*

In other words, Tor can be computed using flat resolutions (not just projective resolutions).

10.6. (Torsion)

Let A be an abelian group (**Z**-module). Put

$$\mathrm{t}(A) = \{a \in A \mid na = 0 \text{ for some } 0 \neq n \in \mathbf{Z}\}.$$

Then $\mathrm{t}(A)$ is a subgroup of A called the **torsion subgroup** of A.

A is **torsion** if $\mathrm{t}(A) = A$ and it is **torsion free** if $\mathrm{t}(A) = 0$. The quotient $A/\mathrm{t}(A)$ is torsion free.

Theorem. *Let A and B be abelian groups.*

(i) $\mathrm{Tor}_1^{\mathbf{Z}}(A, B)$ *is a torsion group.*

(ii) $\mathrm{Tor}_n^{\mathbf{Z}}(A, B) = 0$ *for all* $n \neq 1$.

The torsion functor $\mathrm{t} : \mathbf{Ab} \to \mathbf{Ab}$ sends an abelian group A to its torsion subgroup $\mathrm{t}(A)$ and a group homomorphism $f : A \to B$ to its restriction $\mathrm{t}(f) : \mathrm{t}(A) \to \mathrm{t}(B)$.

Theorem. $\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, \cdot\,) \cong \mathrm{t}\,.$

The results in this section remain valid if **Z** is replaced by an arbitrary integral domain and **Q** with its field of quotients.

11 Inverse limit

The notion of inverse limit is dual to the notion of direct limit (just reverse arrows).

11.1. Let $(I, \leq)$ be a preordered set (viewed as a category as in 9.1). An **inverse system** in the category $\mathcal{C}$ with index set I is a contravariant functor $A : I \to \mathcal{C}$ (equivalently, a functor $A : I^{\mathrm{op}} \to \mathcal{C}$).

Let $A : I \to \mathcal{C}$ be an inverse system. We write A_i for $A(i)$ and α_i^j for the image under A of the unique morphism $i \to j$ whenever $i \leq j$.

A **cone** of A is a pair $(C, \{g_i\})$ where $C \in \mathrm{ob}\,\mathcal{C}$ and the morphisms $g_i : C \to A_i$ satisfy $\alpha_i^j g_j = g_i$ whenever $i \leq j$.

An **inverse limit** of the inverse system A is a cone $(\varprojlim A_i, \{f_i\})$ of A such that for every cone $(C, \{g_i\})$ of A there exists a unique morphism

$\varphi : C \to \varprojlim A_i$ such that $f_i\varphi = g_i$ for all $i \in I$:

$$\begin{tikzcd}
\varprojlim A_i \arrow[dr, "f_j"] \arrow[ddr, "f_i"'] & & C \arrow[ll, dashed, "\exists!\,\varphi"'] \arrow[dl, "g_j"'] \arrow[ddl, "g_i"] \\
& A_j \arrow[d, "\alpha_i^j"] & \\
& A_i &
\end{tikzcd}$$

(An inverse limit is an example of a "limit," which is defined the same way, but with I being an arbitrary category.)

THEOREM. *For any inverse system in R-**mod**, an inverse limit exists.*

11.2. (Examples)

- If the preorder $\leq$ is trivial, then $\varprojlim A_i = \prod_{i\in I} A_i$.
- If $I = \{1, 2, 3\}$ with order $1 \leq 2, 1 \leq 3$, then $\varprojlim A_i$ is called the **pullback** of the morphisms α_1^2 and α_1^3.
- If $f : A \to B$ is a morphism, then $\ker f$ is a pullback of f and $0 : 0 \to B$ and hence an inverse limit.
- Let $\{A_i\}_{i\in I}$ be a family of subsets of a set X and put $i \leq j$ if $A_i \supseteq A_j$. If I is directed, then $\varprojlim A_i = \bigcap A_i$.

11.3. (Inverse limit is right exact)

Let I be a preordered set and let $\mathcal{C}$ be a category. The category $\mathcal{C}^{I^{\mathrm{op}}}$ has inverse systems of I in $\mathcal{C}$ as objects and natural transformations as morphisms. If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^{I^{\mathrm{op}}}$.

Assume that $\mathcal{C}$ has an inverse limit for each inverse system of I in $\mathcal{C}$. Then we get a functor $\varprojlim : \mathcal{C}^{I^{\mathrm{op}}} \to \mathcal{C}$ that sends each inverse system to its inverse limit and sends a morphism of inverse systems to the induced morphism.

Define the **diagonal functor** $\triangle : \mathcal{C} \to \mathcal{C}^{I^{\mathrm{op}}}$ by $(\triangle M)_i = M$ for all i and $\alpha_i^j = 1_M$ for $i \le j$.

THEOREM. *$(\triangle, \varprojlim)$ is an adjoint pair.*

In particular, if $\mathcal{C}$ is abelian, then $\varprojlim$ is left exact.

11.4. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and assume that inverse limits exist in $\mathcal{D}$ and $\mathcal{C}$. For any preordered set I, the functor G induces a functor $G : \mathcal{C}^{I^{\mathrm{op}}} \to \mathcal{D}^{I^{\mathrm{op}}}$ with $G(A)_i = G(A_i)$ and $G(\tau)_i = G(\tau_i)$ for each inverse system A and morphism $\tau : A \to B$. We say that G **preserves inverse limits** if for each preordered set I, $\varprojlim G \cong G \varprojlim$ as functors from $\mathcal{D}^{I^{\mathrm{op}}}$ to $\mathcal{C}$.

THEOREM. *If (F, G) is an adjoint pair for some $F : \mathcal{C} \to \mathcal{D}$, then G preserves inverse limits.*

In particular, if G is a right adjoint and $\mathcal{D}$ and $\mathcal{C}$ are abelian, then G is left exact (since a kernel is an inverse limit).

THEOREM. *The following are equivalent for a functor $G : R$-**mod** $\to$ **Ab**:*

(i) *G preserves inverse limits,*

(ii) *G is left exact and preserves products,*

(iii) *$G \cong \mathrm{Hom}_R(B, \cdot\,)$ for some R-module B,*

(iv) *(F, G) is an adjoint pair for some F :* **Ab** $\to R$-**mod**

The implication (ii)⇒(iii) is known as Watts' theorem.

12 Ext

12.1. We translate some elementary properties of derived functors in general to the Ext notation:

(i) $\mathrm{Ext}_R^n(A, B) = 0$ if $n < 0$,

(ii) if B is injective, then $\operatorname{Ext}_R^n(A, B) = 0$ for all $n \neq 0$ and all A,

(iii) if A is projective, then $\operatorname{Ext}_R^n(A, B) = 0$ for all $n \neq 0$ and all B,

(iv) $\operatorname{Ext}_R^0(A, \cdot\,) \cong \operatorname{Hom}_R(A, \cdot\,)$ (since hom is left exact),

(v) $\operatorname{Ext}_R^0(\,\cdot\,, B) \cong \operatorname{Hom}_R(\,\cdot\,, B)$ (since hom is left exact),

(vi) if $0 \to B' \to B \to B'' \to 0$ is exact, then for each A there is a long exact sequence

$$0 \to \operatorname{Hom}_R(A, B') \to \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A, B'') \to \operatorname{Ext}_R^1(A, B') \to \cdots,$$

(vii) if $0 \to A' \to A \to A'' \to 0$ is exact, then for each B there is a long exact sequence

$$0 \to \operatorname{Hom}_R(A'', B) \to \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A', B) \to \operatorname{Ext}_R^1(A', B) \to \cdots.$$

12.2.

Theorem.

(i) *If* $\{B_i\}_{i \in I}$ *is a family of* R*-modules, then for each* A

$$\operatorname{Ext}_R^n(A, \prod B_i) \cong \prod \operatorname{Ext}_R^n(A, B_i).$$

(ii) *If* $\{A_i\}_{i \in I}$ *is a family of* R*-modules, then for each* B

$$\operatorname{Ext}_R^n(\coprod A_i, B) \cong \prod \operatorname{Ext}_R^n(A_i, B).$$

It is not the case that Ext preserves inverse limits or direct limits (even with the assumption that the preordered set is directed).

12.3.

Theorem.

(i) *An* R*-module* B *is injective if and only if* $\operatorname{Ext}_R^1(A, B) = 0$ *for all* A*.*

(ii) *An* R*-module* A *is projective if and only if* $\operatorname{Ext}_R^1(A, B) = 0$ *for all* B*.*

12.4. (Extension)

Let A and C be R-modules. An **extension** of C by A is an exact sequence

$$\epsilon : \quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Two extensions ϵ and ϵ' of C by A are **equivalent** if there exists a chain map from one to the other that is the identity on A and on C:

$$\begin{tikzcd} \epsilon : & 0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & B \arrow[r] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "1"] & 0 \\ \epsilon' : & 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

(in this case φ is an isomorphism by the five lemma). Denote by $[\epsilon]$ the equivalence class of the extension ϵ under this relation and let $e(C, A)$ denote the set of all equivalence classes of extensions of C by A.

We define an addition on $e(C, A)$. Let

$$\epsilon : \quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad \text{and} \qquad \epsilon' : \quad 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$$

be two extensions of C by A. Let S be the pullback of g and g',

$$S = \{(b, b') \,|\, g(b) = g'(b')\},$$

let

$$D = \{(f(a), -f'(a)) \,|\, a \in A\} \subseteq P,$$

and put $\bar{B} = S/D$.

The **Baer sum** of $[\epsilon]$ and $[\epsilon']$ is the class of the extension

$$\epsilon + \epsilon' : \quad 0 \to A \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} C \to 0,$$

where $\bar{f}(a) = \overline{(f(a), 0)}$ and $\bar{g}(\overline{(b, b')}) = g'(b')$.

An extension is **split** if it is equivalent to the extension

$$0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0,$$

where ι and π are the natural maps. Equivalently, the extension ϵ (as shown above) is split if there exists $h : C \to B$ such that $gh = 1_C$ or if there exists $j : B \to A$ such that $jf = 1_A$.

Theorem. *Under Baer sum $e(C,A)$ is an abelian group isomorphic to $\mathrm{Ext}^1_R(C,A)$. The class of the split extension is the zero element of $e(C,A)$.*

We describe a pair of homomorphisms $\Phi : e(C,A) \to \mathrm{Ext}^1_R(C,A)$ and $\Psi : \mathrm{Ext}^1_R(C,A) \to e(C,A)$, with both compositions equaling the identity map:

- $\Phi([\epsilon]) = \partial(1_A)$, where $\partial : \mathrm{Hom}_R(A,A) \to \mathrm{Ext}^1_R(C,A)$ is the connecting homomorphism.
- $\Psi(x) = [\epsilon]$, where ϵ is the extension obtained as follows: Let
$$0 \to K \to P \to C \to 0$$
be an exact sequence with P projective. The sequence
$$\mathrm{Hom}_R(P,A) \to \mathrm{Hom}_R(K,A) \xrightarrow{\partial} \mathrm{Ext}^1_R(C,A) \to 0$$
is exact, so $x = \partial(\beta)$ for some $\beta : K \to A$. Then ϵ is the bottom row of
$$\begin{tikzcd} & 0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[r, "h"] \arrow[d] & C \arrow[r] \arrow[d, "1"] & 0 \\ \epsilon : & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0, \end{tikzcd}$$
where $B = A \oplus P/S$, $S = \{(\beta(k), -j(k)) \mid k \in K\}$ (i.e., first square is a pushout) and g is induced by $0 : A \to C$ and $h : P \to C$.

Here is a proof of the second statement of the theorem. Let $[\epsilon] \in e(C,A)$. The extension ϵ gives rise to an exact sequence
$$\mathrm{Hom}_R(B,A) \xrightarrow{f^*} \mathrm{Hom}_R(A,A) \xrightarrow{\partial} \mathrm{Ext}^1_R(C,A).$$
If $\Phi([\epsilon]) = 0$ then $1_A \in \ker \partial = \mathrm{im}\, f^*$ so there exists $j : B \to A$ such that $1_A = f^*(j) = jf$ and ϵ is split.

12.5. (Yoneda extension)

The results of the preceding section generalize. Let A and C be objects in an abelian category $\mathcal{A}$. A (Yoneda) n**-extension** of C by A is an exact sequence
$$\epsilon : \quad 0 \to A \to B_n \to B_{n-1} \to \cdots \to B_1 \to C \to 0.$$

Let ϵ and ϵ' be n-extensions of C by A. If there exists a commutative diagram

$$\begin{tikzcd}
\epsilon: & 0 \arrow[r] & A \arrow[r] \arrow[d, "1"'] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & B_1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "1"] & 0 \\
\epsilon': & 0 \arrow[r] & A \arrow[r] & B'_n \arrow[r] & B'_{n-1} \arrow[r] & \cdots \arrow[r] & B'_1 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

we write $\epsilon \to \epsilon'$ (and also $\epsilon' \leftarrow \epsilon$). We write $\epsilon \sim \epsilon'$ if there exist n-extensions $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ such that

$$\epsilon \to \epsilon_1 \leftarrow \epsilon_2 \to \cdots \to \epsilon_m \leftarrow \epsilon'.$$

The relation $\sim$ is an equivalence relation on the class of n-extensions of C by A. We denote by $[\epsilon]$ the class of ϵ and by $\mathrm{Yext}^n_{\mathcal{A}}(C, A)$ the collection of all equivalence classes of n-extensions of C by A. We assume that $\mathrm{Yext}^n_{\mathcal{A}}(C, A)$ is a set (which is the case if $\mathcal{A}$ is a module category).

If $n \geq 2$, the **Baer sum** of $[\epsilon]$ and $[\epsilon']$ is the class of the n-extension

$$\epsilon + \epsilon': \quad 0 \to A \to \bar{B}_n \to B_{n-1} \oplus B'_{n-1} \to \cdots \to B_2 \oplus B'_2 \to \bar{B}_1 \to C \to 0,$$

where $\bar{B}_n$ is the pushout of $A \to B_n$, $A \to B'_n$ and $\bar{B}_1$ is the pullback of $B_1 \to C$, $B'_1 \to C$.

Theorem. *Assume that $\mathcal{A}$ has enough projectives. Under Baer sum, $\mathrm{Yext}^n_{\mathcal{A}}(C, A)$ is an abelian group isomorphic to $\mathrm{Ext}^n_{\mathcal{A}}(C, A)$.*