Introduction to Topology

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Chapter 1. Metric Spaces

1. Definition and Examples.

As the course progresses we will need to review some basic notions about sets and functions. We begin with a little set theory.

Let S be a set. For $A, B \subseteq S$, put

$$A \cup B := \{s \in S \,|\, s \in A \text{ or } s \in B\}$$
$$A \cap B := \{s \in S \,|\, s \in A \text{ and } s \in B\}$$
$$S - A := \{s \in S \,|\, s \notin A\}$$

1.1 Theorem. *Let $A, B \subseteq S$. Then $S - (A \cup B) = (S - A) \cap (S - B)$.*

Exercise 1. Let $A \subseteq S$. Prove that $S - (S - A) = A$.

Exercise 2. Let $A, B \subseteq S$. Prove that $S - (A \cap B) = (S - A) \cup (S - B)$. (Hint: Either prove this directly as in the proof of Theorem 1.1, or just use the statement of Theorem 1.1 together with Exercise 1.)

Exercise 3. Let $A, B, C \subseteq S$. Prove that $A \vartriangle C \subseteq (A \vartriangle B) \cup (B \vartriangle C)$, where $A \vartriangle B := (A \cup B) - (A \cap B)$.

1.2 Definition. A *metric space* is a pair (X, d) where X is a non-empty set, and d is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

(1) $d(x, y) \geq 0$,
(2) $d(x, y) = 0$ if and only if $x = y$,
(3) $d(x, y) = d(y, x)$, and
(4) $d(x, z) \leq d(x, y) + d(y, z)$ ("triangle inequality").

In the definition, d is called the *distance function* (or *metric*) and X is called the *underlying set*.

1.3 Example. For $x, y \in \mathbb{R}$, set $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space.

1.4 Example. Let X be a non-empty set. For $x, y \in X$, set $d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ Then (X, d) is a metric space. (d is called the *discrete metric* on X.)

1.5 Example. Let X be the set of all continuous functions $f : [a, b] \to \mathbb{R}$. For $f, g \in X$, set $d(f, g) = \int_a^b |f(t) - g(t)|\, dt$. Then (X, d) is a metric space.

1.6 Example. Let p be a fixed prime number. For $m, n \in \mathbb{Z}$ set

$$d(m, n) = \begin{cases} 0 & m = n, \\ p^{-t} & m \neq n, \end{cases}$$

where $m - n = p^t k$ with k an integer that is not divisible by p. Then $(\mathbb{Z}, d)$ is a metric space.

Exercise 4. Let X be the collection of the interiors of those rectangles in $\mathbb{R}^2$ having sides parallel to the coordinate axes. For $A, B \in X$, let $d(A, B)$ denote the area of $A \triangle B$. Prove that (X, d) is a metric space. (Hint: Use Exercise 3.)

For sets $X_1, \ldots, X_n$, define

$$\prod_{i=1}^{n} X_i := \{(x_1, \ldots, x_n) \mid x_i \in X_i\}.$$

This set is called the *Cartesian product* of $X_1, \ldots, X_n$. Sometimes it is written $X_1 \times \cdots \times X_n$.

Exercise 5. Let $\mathbb{R}^n := \mathbb{R} \times \cdots \times \mathbb{R}$ (n factors). For $x, y \in \mathbb{R}^n, c \in \mathbb{R}$ set

$$\begin{aligned} x + y &= (x_1 + y_1, \ldots, x_n + y_n), \\ cx &= (cx_1, \ldots, cx_n), \\ x \cdot y &= x_1 y_1 + \cdots + x_n y_n, \\ \|x\| &= (x_1^2 + \cdots + x_n^2)^{1/2} \end{aligned}$$

Prove the following:

a. $x \cdot (y + z) = x \cdot y + x \cdot z$.
b. $(cx) \cdot y = c(x \cdot y)$.
c. $x \cdot y \leq ||x|| \, ||y||$ (Hint: $\big\| \|y\|x - \|x\|y \big\|^2 \geq 0$. Use the fact that $\|z\|^2 = z \cdot z$ together with (a) and (b) to expand the left hand side.)
d. $||x + y|| \leq ||x|| + ||y||$ (Hint: Square both sides and use (c).)

Exercise 6. For $x, y \in \mathbb{R}^n$, set $d(x, y) = \big[\sum_{i=1}^{n} (x_i - y_i)^2\big]^{1/2}$. Prove that $(\mathbb{R}^n, d)$ is a metric space. (The function d is called the *Euclidean metric* on $\mathbb{R}^n$.) (Hint: For the triangle inequality, note that $d(x, z) = \|x - z\|$. Use Exercise 5(d).)

1.7 Theorem. *Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces and let $X = \prod_{i=1}^{n} X_i$. For $x, y \in X$, set $d(x, y) = \max\{d_i(x_i, y_i) \mid 1 \leq i \leq n\}$. Then (X, d) is a metric space.*

1.8 Corollary. *For $x, y \in \mathbb{R}^n$, set $\rho(x, y) = \max\{|x_i - y_i| \mid 1 \leq i \leq n\}$. Then $(\mathbb{R}^n, \rho)$ is a metric space.*

The function ρ is called the *square metric* on $\mathbb{R}^n$.

2. Continuous Functions.

2.1 Definition. Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is *continuous at the point* $a \in X$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X$ satisfies

$$d(x, a) < \delta,$$

then $f(x)$ satisfies

$$d'(f(x), f(a)) < \epsilon.$$

The function f is *continuous* if it is continuous at each point of X.

In the case $X = Y = \mathbb{R}$ (usual metric), we have $d(x, a) = |x - a|$ and $d'(f(x), f(a)) = |f(x) - f(a)|$, so this definition of continuity agrees with the usual definition.

2.2 Example. Given a fixed $c \in Y$, the *constant function* $f : X \to Y$ given by $f(x) = c$ $(x \in X)$ is continuous.

2.3 Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 0 & x = 0, \\ x/|x| & x \neq 0, \end{cases}$ is discontinuous (i.e., not continuous).

2.4 Example. Let (X, d) be a metric space. The *identity function* $f : X \to X$ given by $f(x) = x$ is continuous.

2.5 Example. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1 + x_2$ is continuous where $\mathbb{R}$ has the usual metric and $\mathbb{R}^2$ has the square metric.

Exercise 7. Let $a, b \in \mathbb{R}$. Prove that the linear function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax+b$ is continuous. (Hint: For the case $a = 0$ use Example 2.2.)

Exercise 8. Let (X, d) be the metric space defined in Example 1.5. Prove that the function $F : X \to \mathbb{R}$ given by $F(f) = \int_a^b f(t)\, dt$ is continuous where $\mathbb{R}$ has the usual metric.

Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition* of f and g is the function $g \circ f : X \to Z$ given by $(g \circ f)(x) = g(f(x))$.

2.6 Theorem. *Let (X, d), (Y, d'), (Z, d'') be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

Exercise 9. It is shown in calculus that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous. Assuming this, prove that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^4 + 2x^2 + 1$ is also continuous.

3. Limit of a Sequence.

3.1 Definition. Let (X, d) be a metric space and let $(a_n) = (a_1, a_2, \dots)$ be a sequence of elements of X. An element a of X is called the *limit of the sequence* (a_n) if for each $\epsilon > 0$ there exists a positive integer N such that $d(a_n, a) < \epsilon$ for all $n > N$. In this case we say that the sequence *converges to* a and write $\lim_n a_n = a$.

In the case $X = \mathbb{R}$ (usual metric), we have $d(a_n, a) = |a_n - a|$, so this definition of limit agrees with the usual definition.

Exercise 10. Prove that a sequence can have at most one limit.

3.2 Example. Let $a_n = 1/n \in \mathbb{R}$ (usual metric). Then $\lim_n a_n = 0$.

3.3 Example. Let $(\mathbb{Z}, d)$ be the metric space of Example 1.6 relative to the fixed prime number p. We have $\lim_n p^n = 0$.

Exercise 11. Let (a_n) and (b_n) be sequences in $\mathbb{R}$ (usual metric) and assume that $\lim_n a_n = a$ and $\lim_n b_n = b$. Prove that $\lim_n(a_n + b_n) = a + b$.

Exercise 12. Let (X, d) be the metric space of Example 1.5 and let $f_n \in X$ be given by $f_n(x) = x/n$. Prove that $\lim_n f_n = z$, where z denotes the zero function ($z(x) = 0$ $\forall\, a \leq x \leq b$).

3.4 Theorem. *Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is continuous at the point $a \in X$ if and only if for each sequence (a_n) in X converging to a, we have $\lim_n f(a_n) = f(a)$.*

Exercise 13. Prove that $\lim_n \dfrac{1 + 2n^2 + n^4}{n^4} = 1$ by using only what we have shown in this course. (Hint: Use Exercise 9, Example 3.2, and Theorem 3.4.)

4. Open Sets.

Let $f : X \to Y$ be a function. We say that

f is *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ ($x_i \in X$),

f is *surjective* if for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$,

f is *bijective* if it is both injective and surjective.

Given $A \subseteq X$, the subset

$$f(A) := \{f(a) \,|\, a \in A\}$$

of Y is called the *image of A under f*. Given $B \subseteq Y$, the subset

$$f^{-1}(B) := \{x \in X \,|\, f(x) \in B\}$$

of X is called the *inverse image of B under f*.

If f is bijective, then there exists an inverse function $f^{-1} : Y \to X$ which sends $f(x)$ to x, and in this case $f^{-1}(B)$ coincides with the image of B under f^{-1} as the notation suggests. However, the inverse image of B under f is defined even when the inverse function f^{-1} is not.

Exercise 14. Let $f : X \to Y$ be a function and let $A \subseteq X$, $B \subseteq X$. Prove the following:

a. $A \subseteq f^{-1}(f(A))$ with equality if f is injective.
b. $B \supseteq f(f^{-1}(B))$ with equality if f is surjective.

4.1 Definition. Let (X, d) be a metric space. For $a \in X$ and $\epsilon > 0$, the set

$$B_\epsilon(a) := \{x \in X \,|\, d(x, a) < \epsilon\}$$

is called the *open ball about a of radius ϵ* (or the *ϵ-ball about a*).

The following theorems express the notions of continuity and limit using this new notation.

4.2 THEOREM. *Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is continuous at $a \in X$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$ (or equivalently, $B_\delta(a) \subseteq f^{-1}[B_\epsilon(f(a))]$).*

4.3 THEOREM. *Let (X, d) be a metric space and let (a_n) be a sequence in X. Then $\lim_n a_n = a$ if and only if for each $\epsilon > 0$ there exists a positive integer N such that $a_n \in B_\epsilon(a)$ for all $n > N$.*

For the rest of this section, (X, d) denotes a metric space.

4.4 Definition. A subset U of X is *open* if for each $a \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(a) \subseteq U$.

4.5 Example. The open subsets of $\mathbb{R}$ (usual metric) are just the unions of open intervals (e.g., $(-3, -1) \cup (40/7, \infty)$). The set $[2, 5)$ is not open in $\mathbb{R}$.

4.6 Example. The set of points enclosed by a closed curve (not including the curve itself) is an open subset of $\mathbb{R}^2$ (with either the Euclidean metric or the square metric). The set of points lying outside a closed curve (again, not including the curve) is also open.

Exercise 15. Given $a \in X$ and $\epsilon > 0$, prove that $B_\epsilon(a)$ is open.

Exercise 16. Let (X, d) be the metric space of Example 1.4. Prove that *every* subset of X is open.

4.7 THEOREM. *Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open for each open subset U of Y.*

Let I be a set. Suppose that for each $\alpha \in I$ we have a set A_α. Then $\{A_\alpha\}_{\alpha \in I}$ is called an *indexed family of sets* and I is called the *index set.*

Extending the concepts of union and intersection, we define

$$\cup_\alpha A_\alpha := \{a \,|\, a \in A_\alpha \text{ for some } \alpha \in I\}$$
$$\cap_\alpha A_\alpha := \{a \,|\, a \in A_\alpha \text{ for all } \alpha \in I\}$$

Exercise 17. Let $f : X \to Y$ be a function and let $\{A_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of X.

a. Prove that $f(\cup_\alpha A_\alpha) = \cup_\alpha f(A_\alpha)$.
b. Prove that $f(\cap_\alpha A_\alpha) \subseteq \cap_\alpha f(A_\alpha)$ with equality if f is injective.

4.8 Theorem.

(1) *X is open,*
(2) *$\emptyset$ is open,*
(3) *If $U_1, \dots, U_n \subseteq X$ are open, then so is $U_1 \cap \cdots \cap U_n$,*
(4) *If $\{U_\alpha\}_{\alpha \in I}$ is a family of open subsets of X, then $\cup_\alpha U_\alpha$ is open.*

4.9 Example. For each positive integer n, set $U_n := (-1/n, 1/n) \subseteq \mathbb{R}$. Then $\cap_n U_n = \{0\}$, which is not open. This shows that we can only allow finitely many U_i in part (3) of the Theorem.

Exercise 18. Let U be a nonempty subset of X. Prove that U is open if and only if it equals a union of (possibly infinitely many) open balls.

5. Closed Sets.

Let (X, d) be a metric space.

5.1 Definition. A subset A of X is *closed* if $X - A$ is open.

Riddle: *How is a subset of X different from a door?* Answer: It is possible for a subset of X to be neither closed nor open (e.g., $[0, 1) \subseteq \mathbb{R}$ (usual metric)). It is also possible for a subset of X to be both closed and open (*clopen*) (e.g., $\emptyset$ is clopen, as is X).

5.2 Example. The subset $(-\infty, -3] \cup [-1, 40/7]$ of $\mathbb{R}$ (usual metric) is closed.

5.3 Example. The set of points enclosed by a closed curve (including the curve) is a closed subset of $\mathbb{R}^2$ (either metric), as is the set of points outside a closed curve (including the curve).

Exercise 19. Let A be a subset of X. A point b of X is a *limit point of A* if each open ball about b contains a point of A different from b.

a. Prove that b is a limit point of A only if there exists a sequence (a_n) in A that converges to b.
b. Prove that A is closed if and only if it contains all its limit points.

5.4 Theorem. *A subset A of X is closed if and only if for each sequence (a_n) in A that converges to $a \in X$, we have $a \in A$.*

Exercise 20. Let (X, d) and (Y, d') be metric spaces. Prove that a function $f : X \to Y$ is continuous if and only if $f^{-1}(A)$ is closed for each closed $A \subseteq Y$.

5.5 Theorem.

(1) *X is closed.*
(2) *$\emptyset$ is closed.*
(3) *If $A_1, \dots, A_n \subseteq X$ are closed, then so is $A_1 \cup \cdots \cup A_n$.*
4. *If $\{A_\alpha\}_{\alpha \in I}$ is a family of closed subsets of X, then $\cap_\alpha A_\alpha$ is closed.*

Chapter 2. Topological Spaces

1. Definition and Examples.

1.1 Definition. A *topological space* is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a collection of subsets of X such that

(1) $X \in \mathcal{T}$,
(2) $\emptyset \in \mathcal{T}$,
(3) If $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$,
(4) If $\{U_\alpha\}_{\alpha \in I}$ is an indexed family with $U_\alpha \in \mathcal{T}$ for each $\alpha \in I$, then $\cup_\alpha U_\alpha \in \mathcal{T}$.

The set X is called the *underlying set;* its elements are called *points.* The collection $\mathcal{T}$ is called the *topology* on X; its elements are called *open sets.*

1.2 Example. Let (X, d) be a metric space and let $\mathcal{T}$ be the collection of all open sets (as defined in 4.4 of Chapter 1). Then, according to 4.8 of Chapter 1, $(X, \mathcal{T})$ is a topological space. $\mathcal{T}$ is called the *topology induced by the metric d.*

1.3 Example. Let X be a nonempty set and let $\mathcal{T}$ be the collection of *all* subsets of X. Then $(X, \mathcal{T})$ is a topological space. $\mathcal{T}$ is called the *discrete* topology.

1.4 Example. Let X be a nonempty set and let $\mathcal{T} = \{X, \emptyset\}$. Then $(X, \mathcal{T})$ is a topological space. $\mathcal{T}$ is called the *indiscrete* topology.

1.5 Example. Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then $(X, \mathcal{T})$ is a topological space. (Note that if we remove the set $\{b\}$ from $\mathcal{T}$, then $(X, \mathcal{T})$ is no longer a topological space since in that case $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}$ and 1.1(3) is not satisfied.)

1.6 Example. Let $X = \mathbb{N} := \{1, 2, 3, \ldots\}$ and for each $n \in \mathbb{N}$ set $U_n = \{n, n+1, n+2, \ldots\}$. Let $\mathcal{T} = \{\emptyset, U_n \mid n \in \mathbb{N}\}$. Then $(X, \mathcal{T})$ is a topological space.

1.7 Example. Let X be a nonempty set and let $\mathcal{T} = \{U \subseteq X \mid U = \emptyset \text{ or } |X - U| < \infty\}$. Then $(X, \mathcal{T})$ is a topological space. $\mathcal{T}$ is called the *finite complement* topology.

Exercise 21. Let $(X, \mathcal{T})$ be a topological space and let $S \subseteq X$. Assume that for each $s \in S$ there is an open set U such that $s \in U \subseteq S$. Prove that S is open.

For the rest of the section, $(X, \mathcal{T})$ denotes a topological space.

1.8 Definition. $(X, \mathcal{T})$ is *metrizable* if there exists a metric d on X that induces $\mathcal{T}$. (Recall from Example 1.2 that d *induces* $\mathcal{T}$ if the elements of $\mathcal{T}$ are precisely the open sets as defined for the metric space (X, d). In other words, a subset U of X is in $\mathcal{T}$ if and only if for each $a \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subseteq U$.)

1.9 Example. If a nonempty set X is given the discrete topology $\mathcal{T}$ of Example 1.3, then the toplogical space $(X, \mathcal{T})$ is metrizable.

1.10 Theorem. *If $(X, \mathcal{T})$ is metrizable, then for each $x, y \in X$ with $x \neq y$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

1.11 Example. The topological space of Example 1.5 is not metrizable.

Exercise 22. Examine the topological spaces of Examples 1.4, 1.6, and 1.7, and decide under what circumstances, if any, each is metrizable.

The following exercise shows that it is possible for two *different* metrics on a set to induce the *same* topology.

Exercise 23. Let d and ρ denote the Euclidean metric and the square metric, respectively, on $\mathbb{R}^2$ and let $\mathcal{T}_d$ and $\mathcal{T}_\rho$ denote the induced topologies (see 1.8 for the precise meaning of "induced"). Prove that $\mathcal{T}_d = \mathcal{T}_\rho$.

1.12 Definition. A subset A of X is *closed* if $X - A$ is open.

Exercise 24. Prove that $U \subseteq X$ is open if and only if $X - U$ is closed.

1.13 Theorem.

(1) *X is closed.*
(2) *$\emptyset$ is closed.*
(3) *If $A_1, \ldots, A_n \subseteq X$ are closed, then so is $A_1 \cup \cdots \cup A_n$.*
(4) *If $\{A_\alpha\}_{\alpha \in I}$ is a family of closed subsets of X, then $\cap_\alpha A_\alpha$ is closed.*

1.14 Definition. Let $x \in X$. Any open set containing x is called a *neighborhood* of x.

The following definition generalizes the metric space notion of "limit point" given in Exercise 19.

1.15 Definition. Let S be a subset of X. A point $a \in X$ is a *limit point* of S if every neighborhood of a contains a point of S other than a.

1.16 Theorem. *Let A be a subset of X. Then A is closed if and only if it contains all its limit points.*

2. Closure, Interior, Boundary.

Let $(X, \mathcal{T})$ be a topological space.

2.1 Definition. Let S be a subset of X. The *closure* S^- of S is the intersection of all closed subsets of X containing S:

$$S^- := \bigcap_{\substack{A \supseteq S \\ A,\ \text{closed}}} A.$$

Note that S^- is closed by 1.13(4). In fact, S^- can be thought of as the smallest closed set containing S, for if A is any closed set containing S, then A is one of the sets appearing on the right in the definition of S^- so that $A \supseteq S^-$.

2.2 Example. Let $S = (-1, 1) \subset \mathbb{R}$ (usual topology). Then $S^- = [-1, 1]$. More generally, if $S = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\} \subset \mathbb{R}^n$ (usual topology), then $S^- = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$.

2.2 Example. Let $S = (-1, 1) \subset \mathbb{R}$ (usual topology). Then $S^- = [-1, 1]$. More generally, if $S = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\} \subset \mathbb{R}^n$ (usual topology), then $S^- = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$.

2.3 Theorem. *A subset A of X is closed if and only if $A^- = A$.*

Exercise 25. Let $S, T \subseteq X$.

a. Prove that $(S \cup T)^- = S^- \cup T^-$.
b. Prove that $(S \cap T)^- \subseteq S^- \cap T^-$ and give an example to show that equality need not hold.
c. Prove that $(S^-)^- = S^-$.

2.4 Theorem. *Let S be a subset of X and let S' denote the set of all limit points of S. Then $S^- = S \cup S'$.*

Exercise 26. Let $\mathcal{P}$ be the collection of all subsets of X. Define functions $f, g : \mathcal{P} \to \mathcal{P}$ by $f(S) = S^-$ and $g(S) = X - S$. If we start with a set $S \in \mathcal{P}$ we can apply f and g in succession to form other sets. For instance, if $X = \mathbb{R}$ (usual topology) and $S = (0, 1)$, then

$$\begin{aligned} f(S) &= [0, 1], \\ g(f(S)) &= (-\infty, 0) \cup (1, \infty), \\ f(g(f(S))) &= (-\infty, 0] \cup [1, \infty), \\ g(f(g(f(S)))) &= (0, 1) = S. \end{aligned}$$

a. Prove that by starting with $S \in \mathcal{P}$ and applying f and g in succession (starting with f, and then again starting with g), no more than 14 different subsets of X can ever be obtained.
b. Give an example of an $S \subseteq \mathbb{R}$ (usual topology) for which 14 different subsets are actually obtained.

2.5 Definition. Let S be a subset of X. The *interior* S° of S is the union of all open subsets of X contained in S:

$$S^\circ := \bigcup_{\substack{U \subseteq S \\ U, \text{ open}}} U.$$

Using an argument similar to that used for closure, we have that S° is the largest open subset of X contained in S.

2.6 Example. If $S = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\} \subset \mathbb{R}^n$ (usual topology), then $S^\circ = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$.

2.7 Theorem. *Let S be a subset of X.*

(1) $S^\circ = X - (X - S)^-$.
(2) $S^- = X - (X - S)^\circ$.

Exercise 27. Let $S, T \subseteq X$.

a. Prove that $(S \cap T)^\circ = S^\circ \cap T^\circ$.
b. Prove that $(S^\circ)^\circ = S^\circ$.

(Hint: In both (a) and (b) one could proceed directly, of course, but it would be a lot more fun to use Theorem 2.7 and Exercise 25.) Incidentally, as one might expect judging from Exercise 25(b), it is also true that $(S \cup T)^\circ \supseteq S^\circ \cup T^\circ$, with equality not holding in general.

2.8 Theorem. *If S is a subset of X, then $S^\circ = \{a \in X \mid U \subseteq S$ for some neighborhood U of $a\}$.*

2.9 Definition. Let S be a subset of X. The *boundary* S^b of S is defined by $S^b := S^- \cap (X - S)^-$.

2.10 Example. If $S = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\} \subset \mathbb{R}^n$ (usual topology), then $S^b = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$.

2.11 Theorem. *If S is a subset of X, then $S^b = S^- - S^\circ$.*

Exercise 28. A collection $\{S_1, \ldots, S_n\}$ of subsets of X is a *partition* of X if $X = S_1 \cup \cdots \cup S_n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. Let S be a subset of X. Prove that $\{S^\circ, S^b, (X - S)^\circ\}$ is a partition of X. (Hint: You might find Theorem 2.7 useful.)

3. Continuity and Homeomorphism.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces.

3.1 Definition. A function $f : X \to Y$ is *continuous* if $f^{-1}(V)$ is open for each open $V \subseteq Y$.

According to Theorem 4.7 of Chapter 1, this definition generalizes that of a continuous function between metric spaces.

Exercise 29. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ (usual topology) and an open subset U of $\mathbb{R}$ such that $f(U)$ is not open.

3.2 Theorem. *A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed for each closed $B \subseteq Y$.*

Let $(Z, \mathcal{T}'')$ be another topological space.

3.3 Theorem. *If the functions $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composition $g \circ f : X \to Z$.*

Exercise 30. Prove Theorem 3.3.

3.4 Theorem. *A function $f : X \to Y$ is continuous if and only if $f(S^-) \subseteq f(S)^-$ for each subset S of X.*

Let 1_X denote the identity function on X. So $1_X : X \to X$ is given by $1_X(x) = x$ $(x \in X)$.

3.5 Theorem. *A function $f : X \to Y$ is bijective if and only if there exists a function $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.*

Assume $f : X \to Y$ is bijective. The function g guaranteed by the theorem is unique; it is called the *inverse* of f and is denoted f^{-1}. Note that f^{-1} is also bijective.

3.6 Definition. $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are said to be *homeomorphic* if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we call any such functions f and g *homeomorphisms* and we write $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$ (or just $X \simeq Y$ if the topologies are understood).

According to Theorem 3.5, a homeomorphism is necessarily bijective.

3.7 Example. If (a, b) and (c, d) are two open intervals in $\mathbb{R}$ endowed with the usual topologies, then $(a, b) \simeq (c, d)$.

3.8 Definition. A function $f : X \to Y$ is *open* if $f(U)$ is open for each open $U \subseteq X$.

3.9 Theorem. *A function $f : X \to Y$ is a homeomorphism if and only if it is bijective, continuous, and open.*

Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are homeomorphic and let $f : X \to Y$ be a homeomorphism. Since f is a bijection, we can think of it as a renaming function: the element x of X gets renamed $f(x) \in Y$. In particular, we can think of the set Y as just the set X with elements renamed. This renaming function is compatible with the topologies in the sense that U is in $\mathcal{T}$ if and only if the renamed elements of U (namely the elements of $f(U)$) form an element of $\mathcal{T}'$. This implies that any property $(X, \mathcal{T})$ has, that can be expressed entirely in terms of open sets, $(Y, \mathcal{T}')$ must also have.

3.10 Theorem. *If $(X, \mathcal{T})$ is metrizable and $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$, then $(Y, \mathcal{T}')$ is also metrizable.*

3.11 Example. Let $(X, \mathcal{T})$ be the topological space of Example 1.5 and let $\mathcal{T}'$ denote the discrete topology on X. Then $(X, \mathcal{T}) \not\simeq (X, \mathcal{T}')$.

Incidentally, we can use this example to show that Theorem 3.9 is not valid if we remove the word "open." In other words, a bijective continuous function need not be a homeomorphism. Indeed, if we view the identity function 1_X as a function from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$, then it is bijective and continuous (but not open). Yet the example shows that it cannot possibly be a homeomorphism.

3.12 Definition. $(X, \mathcal{T})$ is *Hausdorff* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

According to Theorem 1.10, any metrizable space is Hausdorff. The topological space of Example 1.5 is not Hausdorff.

Exercise 31. Prove that if $(X, \mathcal{T})$ is Hausdorff and $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$, then $(Y, \mathcal{T}')$ is also Hausdorff. (Warning: There exist Hausdorff spaces that are not metrizable, so Theorem 3.10 is of no use here. However, the techniques used in its proof might be useful.)

4. Subspaces.

Let $(X, \mathcal{T})$ be a topological space, let Y be a nonempty subset of X, and set $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$.

4.1 Theorem. *$(Y, \mathcal{T}_Y)$ is a topological space.*

$(Y, \mathcal{T}_Y)$ is said to be a *subspace* of $(X, \mathcal{T})$ and $\mathcal{T}_Y$ is called the *topology* on Y *induced* by $\mathcal{T}$. The elements of $\mathcal{T}_Y$ are said to be *open relative to* Y.

4.2 Example. Let $X = \mathbb{R}$ (usual topology) and let $Y = [0, 2] \subset \mathbb{R}$. Then $[0, 1)$ is open relative to Y since $[0, 1) = (-1, 1) \cap Y$ and $(-1, 1)$ is open in X.

4.3 Theorem. *If $\emptyset \neq Z \subseteq Y \subseteq X$, then $(\mathcal{T}_Y)_Z = \mathcal{T}_Z$.*

Just as we say that a subset of Y is open relative to Y if it is of the form $U \cap Y$ for some open $U \subseteq X$, we say that a subset of Y is *closed relative to* Y if it is of the form $A \cap Y$ for some closed $A \subseteq X$. Now, in the topological space $(Y, \mathcal{T}_Y)$ we already have a notion of a closed subset, namely, $B \subseteq Y$ is closed if $Y - B$ is in $\mathcal{T}_Y$. The following theorem says that these two notions coincide.

4.4 Theorem. *A subset of the topological space $(Y, \mathcal{T}_Y)$ is closed if and only if it is closed relative to Y.*

Exercise 32. Assume Y is an *open* subset of X and let $V \subseteq Y$. Prove that $V \in \mathcal{T}_Y$ if and only if $V \in \mathcal{T}$.

Exercise 33. Prove that if $(X, \mathcal{T})$ is Hausdorff, then $(Y, \mathcal{T}_Y)$ is also Hausdorff.

5. Products.

Before discussing products, we need some preliminaries.

In a metric space (X, d) we can use the open balls to define a topology on X (cf. Example 1.2). Actually, the metric d is more than we really need for this construction. It turns out that if we have a collection $\mathcal{B}$ of subsets of X that merely behaves enough like the collection of open balls, then we can use $\mathcal{B}$ to define a topology on X. This is made more precise in the following theorem.

5.1 Theorem. *Let X be a nonempty set and let $\mathcal{B}$ be a collection of subsets of X such that*

(1) *Each element of X is contained in some element of $\mathcal{B}$,*
(2) *Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.*

Let $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ be the collection of all subsets U of X having the property that for each $a \in U$ there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$. Then $(X, \mathcal{T})$ is a topological space.

The collection $\mathcal{B}$ is called a *basis* for the topology $\mathcal{T}$, and $\mathcal{T}$ is said to be *induced* by $\mathcal{B}$. Note that $\mathcal{B} \subseteq \mathcal{T}$, that is, every basis element is automatically open.

5.2 Example. Let (X, d) be a metric space and let $\mathcal{B} = \{B_\epsilon(a) \mid a \in X, \epsilon > 0\}$. Then $\mathcal{B}$ is a basis for the topology induced by d.

5.3 Example. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} = \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$. In other words, $\mathcal{T}_{\mathcal{T}} = \mathcal{T}$.

5.4 Theorem. *Let the notation be as in 5.1. A subset U of X is open (i.e., an element of $\mathcal{T}$) if and only if it is a union of elements of $\mathcal{B}$.*

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and set $X = X_1 \times X_2 := \{(x_1, x_2) \mid x_i \in X_i\}$. Define

$$\mathcal{B} = \{U_1 \times U_2 \mid U_i \in \mathcal{T}_i\}.$$

It is easily checked that $\mathcal{B}$ satisfies conditions (1) and (2) of Theorem 5.1. Therefore, if we let $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ be the induced topology, we get a topological space $(X, \mathcal{T})$ called the (*Cartesian*) *product* of the topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. $\mathcal{T}$ is called the *product topology.*

5.5 Example. Let $X_1 = X_2 = \mathbb{R}$ (usual topology) and let $X = X_1 \times X_2 = \mathbb{R}^2$. If we let $\mathcal{T}_d$ denote the topology induced by the Euclidean metric d on X and let $\mathcal{T}$ denote the product topology on X, then $\mathcal{T} = \mathcal{T}_d$. In other words, the product topology on $\mathbb{R}^2$ is just the usual topology.

5.6 Theorem. *For any $b \in X_2$, we have $X_1 \times \{b\} \simeq X_1$ where $X_1 \times \{b\} = \{(x_1, b) \mid x_1 \in X_1\} \subseteq X$ is given the subspace topology. (Similarly, $\{a\} \times X_2 \simeq X_2$ for each $a \in X_1$.)*

Exercise 34. Prove that the function $\pi_1 : X \to X_1$ given by $\pi_1((x_1, x_2)) = x_1$ is continuous.

Exercise 35. Prove that a topological space $(X, \mathcal{T})$ is Hausdorff if and only if the "diagonal" $\Delta := \{(x, x) \mid x \in X\}$ is closed in the product $X \times X$.

Chapter 3. Connected Spaces

1. Definition and Examples.

Let $(X, \mathcal{T})$ be a topological space.

1.1 Definition. $(X, \mathcal{T})$ is *not connected* if there exist open subsets U and V of X such that

(1) $U, V \neq \emptyset$,
(2) $U \cap V = \emptyset$,
(3) $U \cup V = X$.

Otherwise, $(X, \mathcal{T})$ is *connected*.

A nonempty subset Y of X is *not connected* (resp., *connected*) if $(Y, \mathcal{T}')$ is not connected (resp., connected) where $\mathcal{T}'$ is the subspace topology.

1.2 Example. The subset $[0,1) \cup (2,3)$ of $\mathbb{R}$ (usual topology) is not connected, whereas any interval in $\mathbb{R}$ is connected (more about this in the next section).

1.3 Theorem. *$(X, \mathcal{T})$ is connected if and only if X and $\emptyset$ are the only clopen subsets of X.*

Exercise 36. Prove Theorem 1.3.

Let $(Y, \mathcal{T}')$ be another topological space.

1.4 Theorem. *Let $f : X \to Y$ be continuous. If S is a connected subset of X, then $f(S)$ is connected.*

1.5 Corollary. *If $(X, \mathcal{T})$ is connected and $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$, then $(Y, \mathcal{T}')$ is connected.*

1.6 Theorem. *If S is a connected subset of X and $S \subseteq T \subseteq S^-$, then T is connected.*

1.7 Lemma. *Let $Y = \{0, 1\}$ (discrete topology). $(X, \mathcal{T})$ is connected if and only if the only continuous functions $f : X \to Y$ are the constant functions (i.e., $f(X) = \{0\}$ or $f(X) = \{1\}$).*

1.8 Theorem. *If $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are connected, then so is $X \times Y$ (with the product topology).*

Exercise 37. Let $S, K \subseteq X$ with S connected, K clopen, and $S \cap K \neq \emptyset$. Prove that $S \subseteq K$.

Exercise 38. Let S and T be connected subsets of X with $S \cap T \neq \emptyset$. Prove that $S \cup T$ is connected. (Hint: Use Theorem 1.3 and Exercise 37.)

2. Applications to the real line.

2.1 Definition. A subset S of $\mathbb{R}$ is an *interval* if whenever $a < b$ are elements of S and $x \in \mathbb{R}$ satisfies $a < x < b$, then $x \in S$.

2.2 Example. $(-\infty, 1]$, $(1, 2)$, $\{3\}$, and $\mathbb{R}$ are all intervals.

2.3 Theorem. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

2.4 Theorem. (Intermediate-Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For each number r between $f(a)$ and $f(b)$, there exists a number $c \in [a, b]$ such that $f(c) = r$.*

2.5 Theorem. (Brouwer's Fixed-Point Theorem) *If $f : [0, 1] \to [0, 1]$ is continuous, then $f(c) = c$ for some $c \in [0, 1]$.*

3. Path-Connected Spaces.

Let $(X, \mathcal{T})$ be a topological space.

3.1 Definition. Let $a, b \in X$. A *path* from a to b is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. The image $f([0, 1])$ of a path f is a *curve.*

3.2 Example. The function $f : [0, 1] \to \mathbb{R}^2$ (usual topology) given by $f(t) = (2t - 1, (2t - 1)^2)$ is a path from $(-1, 1)$ to $(1, 1)$.

Let $(Y, \mathcal{T}')$ be another topological space.

3.3 Lemma. (Pasting Lemma) *Let A_1, A_2 be closed subsets of X, let $f_i : A_i \to Y$ ($i = 1, 2$) be continuous functions and assume $f_1(a) = f_2(a)$ for every $a \in A_1 \cap A_2$. Set $A := A_1 \cup A_2$. The function $f : A \to Y$ defined by*

$$f(a) = \begin{cases} f_1(a), & a \in A_1 \\ f_2(a), & a \in A_2 \end{cases}$$

is continuous.

Exercise 39. A relation $\sim$ on X is an *equivalence relation* if for all $a, b, c \in X$

(1) $a \sim a$ (reflexive property),
(2) $a \sim b$ implies $b \sim a$ (symmetric property),
(3) $a \sim b$, $b \sim c$ implies $a \sim c$ (transitive property).

For $a \in X$, the set $[a] := \{x \in X \mid x \sim a\}$ is called the *equivalence class* of a. The distinct equivalence classes form a partition of X.

For $a, b \in X$, set $a \sim b$ if there exists a path from a to b. Prove that $\sim$ is an equivalence relation on X. (The equivalence classes are called *path components.*) (Hint: Use 3.3 for the transitive property.)

3.4 Definition. $(X, \mathcal{T})$ is *path-connected* if for each $a, b \in X$ there exists a path from a to b. A nonempty subset Y of X is *path-connected* if $(Y, \mathcal{T}')$ is path-connected where $\mathcal{T}'$ is the subspace topology.

3.5 Example. $\mathbb{R}$ (usual topology) is path-connected, for if $a, b \in \mathbb{R}$, then $f : [0,1] \to \mathbb{R}$ given by $f(t) = a + (b-a)t$ is a path from a to b. More generally, $\mathbb{R}^n$ is path-connected for any n (see Theorem 3.9 below).

3.6 Example. The topological space $\{0,1\}$ (discrete topology) is not path-connected by Lemma 1.7.

3.7 Theorem. *If $(X, \mathcal{T})$ is path-connected and $f : X \to Y$ is continuous and surjective, then Y is path-connected.*

Exercise 40. Prove Theorem 3.7 and give an example to show that the statement is false if f is not surjective.

3.8 Corollary. *If $(X, \mathcal{T})$ is path-connected and $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$, then $(Y, \mathcal{T}')$ is path-connected.*

3.9 Theorem. *If $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are path-connected, then so is $X \times Y$ (product topology).*

3.10 Theorem. *If $(X, \mathcal{T})$ is path-connected, then it is connected.*

The following example shows that the converse to Theorem 3.10 does not hold.

3.11 Example. The subspace $Y := \{(x, \sin(1/x) \,|\, x > 0\} \cup \{(0,0)\}$ of $\mathbb{R}^2$ (usual topology) is connected but not path-connected.

Chapter 4. Compact Spaces

1. Definition and Examples.

Let $(X, \mathcal{T})$ be a topological space.

1.1 Definition. Let S be a subset of X. A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of subsets of X is a *covering* of S if $\cup_\alpha U_\alpha \supseteq S$. The collection $\mathcal{U}$ is *open* if each U_α is open and it is *finite* if I is finite. If $J \subseteq I$, then $\mathcal{U}' = \{U_\alpha\}_{\alpha \in J}$ is a *subcollection* of $\mathcal{U}$.

1.2 Example. For each $n \in \mathbb{N}$, set $U_n = (-1 + 1/n, 1 - 1/n) \subseteq \mathbb{R}$. Then $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is an open covering of $(-1, 1) \subset \mathbb{R}$. $\mathcal{U}$ is also an open covering of $[-1/2, 1/2]$.

1.3 Definition. X is *compact* if each open covering of X has a finite subcollection that is also a covering of X. A nonempty subset S of X is *compact* if $(S, \mathcal{T}_S)$ is compact ($\mathcal{T}_S =$ subspace topology).

1.4 Theorem. *A nonempty subset S of X is compact if and only if each covering of S consisting of open subsets of X has a finite subcollection that is also a covering of S.*

1.5 Example. In Example 1.2, $(-1, 1)$ is not compact since no finite subcollection of $\mathcal{U}$ is a covering. Incidentally, the finite subcollection $\{U_1, U_2, U_3\}$ of $\mathcal{U}$ is a covering of $[-1/2, 1/2]$, but this does not prove, of course, that $[-1/2, 1/2]$ is compact, since to prove this one would have to show that *every* open covering has a finite subcollection that is a covering. It will be shown in Section 3 that $[-1/2, 1/2]$ *is* compact, nevertheless.

Exercise 41. Prove that every finite subset of X is compact.

1.6 Theorem. *X is compact if and only if for each collection $\{A_\alpha\}_{\alpha \in I}$ of closed subsets of X satisfying $\cap_{\alpha \in J} A_\alpha \neq \emptyset$ for every finite $J \subseteq I$, we have $\cap_{\alpha \in I} A_\alpha \neq \emptyset$.*

Let $(Y, \mathcal{T}')$ be another topological space.

1.7 Theorem. *Let $f : X \to Y$ be continuous. If S is a compact subset of X, then $f(S)$ is compact.*

Exercise 42. Prove Theorem 1.7. (Hint: Use Theorem 1.4.)

1.8 Theorem. *If $(X, \mathcal{T})$ is compact and $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$, then $(Y, \mathcal{T}')$ is compact.*

2. Compactness and Closed Sets.

Let $(X, \mathcal{T})$ be a topological space.

2.1 Theorem. *If $(X, \mathcal{T})$ is compact and A is a closed subset of X, then A is compact.*

Of course, if $(X, \mathcal{T})$ is not compact, then a closed subset of X need not be compact, since, for instance, X itself is closed.

2.2 THEOREM. *If $(X, \mathcal{T})$ is Hausdorff and S is a compact subset of X, then S is closed.*

Exercise 43. Give an example to show that the Hausdorff assumption in Theorem 2.2 cannot be removed.

Let $(Y, \mathcal{T}')$ be another topological space.

2.3 THEOREM. *If $(X, \mathcal{T})$ is compact, $(Y, \mathcal{T}')$ is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

3. Products of Compact Spaces.

Let $(X, \mathcal{T})$ be a topological space.

3.1 LEMMA. *Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Assume that every collection of elements of $\mathcal{B}$ that is a covering of X has a finite subcollection that is also a covering of X. Then $(X, \mathcal{T})$ is compact.*

3.2 THEOREM. *If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are compact topological spaces, then their product $X = X_1 \times X_2$ is compact.*

If $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ is a family of topological spaces, then there is a natural way to define the product $X = \prod_\alpha X_\alpha$ and a (less natural) way to define a topology $\mathcal{T}$ on X such that if each $(X_\alpha, \mathcal{T}_\alpha)$ is compact, then $(X, \mathcal{T})$ is compact (Tychonoff's Theorem).

3.3 THEOREM. *Any closed interval $[a, b] \subset \mathbb{R}$ (usual topology) is compact.*

3.4 Definition. A subset S of $\mathbb{R}^2$ is *bounded* if $S \subseteq [-b, b] \times [-b, b]$ for some $b \in \mathbb{N}$.

3.5 THEOREM. (Heine-Borel) *A subset S of $\mathbb{R}^2$ (usual topology) is compact if and only if it is closed and bounded.*

One can use a definition similar to 3.4 to define a bounded subset of $\mathbb{R}^n$ for any $n \in \mathbb{N}$. The Heine-Borel theorem is valid in this more general setting, as well.

Chapter 5. Algebraic Topology

1. Fundamental Group.

Let $(X, \mathcal{T})$ be a topological space. Set $I = [0,1] \subset \mathbb{R}$ and $I^2 = I \times I$. Fix $a, b \in X$.

1.1 Definition. Two paths f and g from a to b are *homotopic* (written $f \sim g$) if there exists a continuous function $H : I^2 \to X$ such that for all $t, u \in I$ we have

$$\begin{aligned} H(t,0) &= f(t), \\ H(t,1) &= g(t), \\ H(0,u) &= a, \\ H(1,u) &= b. \end{aligned}$$

Intuitively, f and g are homotopic if f can be continuously deformed to g.

1.2 Theorem. *$\sim$ is an equivalence relation on the set of all paths from a to b.*

The equivalence class of f is denoted $[f]$.
If f and g are two paths from a to itself, then the function $f \cdot g : I \to X$ given by

$$(f \cdot g)(t) := \begin{cases} f(2t) & 0 \leq t \leq 1/2, \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

is also a path from a to itself.
Let $\pi(X, a) := \{[f] \mid f \text{ is a path from } a \text{ to itself}\}$. For $[f], [g] \in \pi(X, a)$, set $[f] \cdot [g] := [f \cdot g]$.

1.3 Theorem. *$(\pi(X, a), \cdot)$ is a group.*

The group $(\pi(X, a), \cdot)$ is called the *fundamental group* of X with *base point* a.

1.4 Example. $\pi(\mathbb{R}^2, a) = \{[e]\}$ for any $a \in \mathbb{R}^2$.

1.5 Example. $\pi(\mathbb{R}^2 \backslash \{(0,0)\}, a) \cong \mathbb{Z}$ (under addition), where $\mathbb{R}^2 \backslash \{(0,0)\}$ is viewed as a subspace of $\mathbb{R}^2$ and a is any element of $\mathbb{R}^2 \backslash \{(0,0)\}$.

1.6 Theorem. *Let $a, b \in X$ and assume that there exists a path from a to b. Then $\pi(X, a) \cong \pi(X, b)$.*

Let $(Y, \mathcal{T}')$ be another topological space.

1.7 Theorem. *Assume $(X, \mathcal{T}) \simeq (Y, \mathcal{T}')$. If $a \in X$, $b \in Y$, and $\varphi : X \to Y$ is a homeomorphism such that $\varphi(a) = b$, then $\varphi_* : \pi(X, a) \to \pi(Y, b)$ given by $\varphi_*([f]) = [\varphi \circ f]$ is an isomorphism.*

1.8 Example. $\mathbb{R}^2 \not\simeq \mathbb{R}^2 \backslash \{(0,0)\}$.