1 System of linear equations

1.1 Two equations in two unknowns

The following is a system of two linear equations in the two unknowns x and y:

$$\begin{aligned} x - y &= 1 \\ 3x + 4y &= 6. \end{aligned}$$

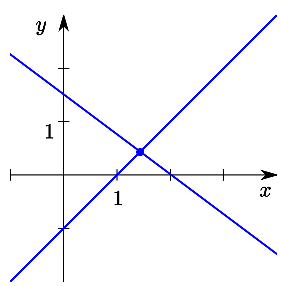
A **solution** to the system is a pair (x, y) of numbers that satisfy both equations. Each of these equations represents a line in the xy-plane, so a solution is a point in the intersection of the lines.

1.1.1 Example (Unique solution)

Sketch the following lines and then solve the system to find the point(s) of intersection:

$$\begin{aligned} x - y &= 1 \\ 3x + 4y &= 6. \end{aligned}$$

Solution We sketch the lines by first finding their x- and y-intercepts. In the first equation, setting $y = 0$ we get $x = 1$, which is the x-intercept, and setting $x = 0$ we get $y = -1$, which is the y-intercept. Similarly, the second line has x-intercept 2 and y-intercept $3/2$. Here is the sketch:



Next, we solve the system. Adding -3 times the first equation to the second gets the x's to drop out:

$$\begin{aligned} -3(\ x - y &= 1) \\ 3x + 4y &= 6 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x - y &= 1 \\ 7y &= 3. \end{aligned}$$

The second equation gives $y = 3/7$ and then the first equation gives $x = 10/7$. Therefore, the lines intersect in the point $(10/7, 3/7)$ (this answer seems reasonable in view of the sketch). □

A system of two linear equations in two unknowns need not always have a unique solution. If the lines that the equations represent are coincident (i.e., the same), then the solution includes every point on the line so there are infinitely many solutions. On the other hand, if the equations represent parallel but not coincident lines, then there is no solution. The following examples illustrate these two possibilities.

1.1.2 Example (Infinitely many solutions)

Solve the following system:

$$\begin{aligned} -x + \ 4y &= \ 2 \\ 3x - 12y &= -6. \end{aligned}$$

Solution Adding 3 times the first equation to the second gets the x's to drop out:

$$\begin{aligned} 3(-x + \ 4y &= \ 2) \\ 3x - 12y &= -6 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} -x + 4y &= 2 \\ 0 &= 0. \end{aligned}$$

The second equation $0 = 0$ places no constraints on x and y so it can be ignored. Therefore, a point (x, y) satisfies the system if and only if it satisfies the first equation, that is, if and only if it is a point on the line $-x+4y = 2$. (The slope-intercept form of both lines is $y = \frac{1}{4}x + \frac{1}{2}$, so the lines are actually coincident.)

We express the solution set by introducing a parameter. If $y = t$, then the equation gives $x = 4t - 2$, so the solution set is

$$\{(4t - 2, t) \mid t \in \mathbf{R}\}.$$

This is read "the set of all points $(4t - 2, t)$ such that t is in the set of real numbers." □

1.1.3 Example (No solution)

Solve the following system:

$$\begin{aligned} x + \ y &= \ 1 \\ 2x + 2y &= -2 \end{aligned}$$

Solution Adding -2 times the first equation to the second gets the x's to drop out:

$$\begin{aligned} -2(x + \ y &= \ 1) \\ 2x + 2y &= -2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x + y &= 1 \\ 0 &= -4. \end{aligned}$$

The equation $0 = -4$ is never satisfied no matter what x and y are. Therefore the solution set is $\emptyset$ (empty set).

(The slope-intercept forms of the lines are

$$\begin{aligned} y &= -x + 1 \\ y &= -x - 1. \end{aligned}$$

The lines have the same slope (-1), so they are parallel. They have different y-intercepts (1 and -1), so they are not coincident.) □

We have seen that a system of two linear equations in two unknowns can have a unique solution, infinitely many solutions, or no solution. It turns out that the same is true no matter how many equations or how many variables there are.

1.2 Three equations in three unknowns

Here is a system of three linear equations in the three unknowns x_1, x_2, and x_3:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

Each of these equations represents a plane in space, so a solution is a point in the intersection of the three planes.

In calculus, the letters x, y, and z are used for the spacial variables, but we use subscripts here to help prepare the way for working with any number of variables.

1.2.1 Example Solve the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

Solution The method is a generalization of that for finding the intersection of two lines (see Section 1.1). Add -2 times the first equation to the second equation in order to cancel the x_1-term:

$$\begin{aligned} -2(\ x_1 - 2x_2 + 3x_3 &= 1) \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ x_2 - x_3 &= -2 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

Then add the first equation to the third equation, again in order to cancel the x_1-term:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= \ \ 1 \\ x_2 - \ \ x_3 &= -2 \\ -x_1 + 4x_2 - \ \ x_3 &= -1 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_1 - 2x_2 + 3x_3 &= \ \ 1 \\ x_2 - \ \ x_3 &= -2 \\ 2x_2 + 2x_3 &= \ \ 0 \end{aligned}$$

Then add -2 times the second equation to the third equation in order to cancel the x_2-term:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= \ \ 1 \\ -2(\qquad x_2 - \ \ x_3 &= -2) \\ 2x_2 + 2x_3 &= \ \ 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_1 - 2x_2 + 3x_3 &= \ \ 1 \\ x_2 - \ \ x_3 &= -2 \\ 4x_3 &= \ \ 4 \end{aligned}$$

The last equation shows that $x_3 = 1$. The other unknowns are determined using a process called "back substitution": Now that we know that $x_3 = 1$, we use the second equation

$$x_2 - \ \ x_3 = -2 \quad \Rightarrow \quad x_2 - (1) = -2 \quad \Rightarrow \quad x_2 = -1$$

and then the first equation

$$x_1 - 2x_2 + 3x_3 = 1 \quad \Rightarrow \quad x_1 - 2(-1) + 3(1) = 1 \quad \Rightarrow \quad x_1 = -4.$$

Therefore, the three planes intersect in the point $(-4, -1, 1)$. The solution set is $\{(-4, -1, 1)\}$. □

This solution illustrates an algorithm for finding the solution to a system of equations called **Gaussian elimination** (after the mathematician Carl Friedrich Gauss).

A system of three linear equations in three unknowns need not have a unique solution. The planes that they represent might be coincident, or they might intersect in a line, either case giving infinitely many solutions. On the other hand, the planes might be parallel but not all coincident, in which case there would be no solution.

A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are numbers.

> Theorem. *A system of linear equations has one of the following:*
>
> - *a unique solution,*
> - *infinitely many solutions,*
> - *no solution.*

In Section 1.6 we obtain a general procedure for writing the solutions to a system of linear equations. But first, we introduce a way of writing systems that will reduce the amount of writing we have to do.

1.3 Augmented matrix of system

The **augmented matrix** of the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

is the matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 5 & 0 \\ -1 & 4 & -1 & -1 \end{array}\right].$$

This is just the array of numbers appearing in the system.

A "matrix" is a rectangular array of numbers. The modifier "augmented" is used here to indicate that the matrix of numbers to the left of the equality signs has been augmented (added on to) by the matrix of numbers to the right.

The augmented matrix of a system provides a way to avoid unnecessary writing when working with the system.

In the solution to the following example, we refer to the "rows" of the matrix. These are the horizontal lists of numbers in the matrix. They are numbered starting from the top.

1.3.1 Example Use the associated augmented matrix to solve the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

Solution The first half of the solution is essentially a repetition of the solution in the preceding example (1.2.1) with the letters x_1, x_2, and x_3, as well as the equality sign, suppressed. We have

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 5 & 0 \\ -1 & 4 & -1 & -1 \end{array}\right]\begin{matrix} -2 \quad 1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 2 & 0 \end{array}\right]\begin{matrix} \\ -2 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 4 & 4 \end{array}\right]\begin{matrix} \\ \\ 1/4 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The steps that we applied in the earlier example are indicated here using numbers and arrows. For instance the first two steps are "add -2 times the first row to the second row" and "add 1 times the first row to the third row." It is useful to keep in mind that the arrow always points to the row that is being changed.

The symbol $\sim$ we use to connect the matrices is read "is row equivalent to" (see Section 1.5).

At this point we could return to the usual notation with x_1, x_2, and x_3 to get

$$\begin{aligned} 1x_1 - 2x_2 + 3x_3 &= 1 \\ 0x_1 + 1x_2 - 1x_3 &= -2 \\ 0x_1 + 0x_2 + 1x_3 &= 1 \end{aligned}$$

or just

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ x_2 \;\; - x_3 &= -2 \\ x_3 &= 1 \end{aligned}$$

and then solve the system by using back substitution as before. Instead, we continue to work with the augmented matrix until the solution becomes obvious:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{matrix} \\ \\ 1 \quad -3 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{matrix} \\ 2 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore,

$$\begin{aligned} x_1 \qquad\qquad &= -4 \\ x_2 \qquad &= -1 \\ x_3 &= 1 \end{aligned}$$

and we get the solution $\{(-4,-1,1)\}$ as before. □

1.4 Row operations

The operations we have been using to reduce a system of equations to one where the solution is apparent are called "row operations." We list them here:

> ROW OPERATIONS.
>
> I. interchange two rows,
>
> II. multiply a row by a nonzero number,
>
> III. add a multiple of one row to another row,
>
> (IV.) add a multiple of one row to a nonzero multiple of another row.

The row operations of type I, II, and III are the **elementary row operations**.

The row operation of type (IV) is just a combination of types II and III (hence the parentheses). It is useful for avoiding fractions as the following example illustrates:

1.4.1 Example Given the augmented matrix

$$\left[\begin{array}{cc|c} 7 & 4 & 1 \\ 3 & 2 & 1 \end{array}\right],$$

create a zero where the 3 is by

(a) using a type (IV) row operation,

(b) using only a type (III) row operation.

Solution (a) Using a type (IV) row operation, we have

$$\left[\begin{array}{cc|c} 7 & 4 & 1 \\ 3 & 2 & 1 \end{array}\right] \begin{matrix} -3 \\ 7 \end{matrix}\Big) \quad \sim \quad \left[\begin{array}{cc|c} 7 & 4 & 1 \\ 0 & 2 & 4 \end{array}\right].$$

(b) Using only a type (III) row operation, we have

$$\left[\begin{array}{cc|c} 7 & 4 & 1 \\ 3 & 2 & 1 \end{array}\right] \begin{matrix} -\frac{3}{7} \\ \end{matrix}\Big) \quad \sim \quad \left[\begin{array}{cc|c} 7 & 4 & 1 \\ 0 & \frac{2}{7} & \frac{4}{7} \end{array}\right].$$

□

Applying a row operation to an augmented matrix is in effect making a change to the corresponding system of equations. It is a fact that the listed row operations do not change the solution set of the system (i.e., a solution to the old system is also a solution to the new system and vice versa). This justifies our method for solving a system of equations, which is to apply row operations until the solution becomes apparent.

1.5 Row echelon form

The method shown for solving a system of linear equations involves applying row operations to the corresponding augmented matrix in order to put it first in "row echelon form" and then finally in "*reduced* row echelon form."

Here is a matrix in "row echelon form" :

$$\begin{bmatrix} 3 & -1 & 2 & 0 & 7 & -5 \\ 0 & 0 & 4 & 8 & -2 & 0 \\ 0 & 0 & 0 & 6 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first nonzero entry in each nonzero row is called that row's **pivot** entry. So the first row has pivot entry 3, the second row has pivot entry 4, and the third row has pivot entry 6.

Row echelon form.

A matrix is in **row echelon form** (abbreviated REF), if

(a) its nonzero rows come before its zero rows,

(b) each of its pivot entries is to the right of the pivot entry in the row above (if any).

Roughly speaking, a matrix is in row echelon form (REF) if the first nonzero entries of the rows form a stair step pattern as shown, with just zeros below and with each step of height one.

For example, the first matrix below is in row echelon form, but the second matrix is not (due to the step of height two):

$$\begin{bmatrix} 2 & -3 & 0 & 4 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(REF)}$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & -1 \\ 0 & -8 & 6 \end{bmatrix} \qquad \text{(not REF)}$$

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **row equivalent** (written $\mathbf{A} \sim \mathbf{B}$) if $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying one or more elementary row operations. Although it is not apparent from the definition, it follows from the reversibility of row operations that $\mathbf{A} \sim \mathbf{B}$ if and only if $\mathbf{B} \sim \mathbf{A}$.

Every matrix is row equivalent to a matrix in row echelon form (REF).

1.5.1 Example Find a matrix in row echelon form (REF) that is row-equivalent to the matrix

$$\begin{bmatrix} 0 & 0 & 6 & 10 & -1 \\ 3 & 1 & -2 & -5 & -3 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix}.$$

Solution The first row starts with a 0. No matter what row operations we apply, we cannot make all of the entries below that 0 also 0's. Therefore, in order to get the desired stair step pattern, we need to interchange the first row with one of the other rows (any will do). After that, we use each pivot entry to create 0's below that entry (keeping a watch for any time we can make the numbers smaller by dividing out a number):

$$
\begin{bmatrix} 0 & 0 & 6 & 10 & -1 \\ 3 & 1 & -2 & -5 & -3 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix} \circlearrowleft
\sim \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix} \begin{matrix} -2 \quad 1 \\ \\ \\ \\ \end{matrix}
$$

$$
\sim \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 4 & 1 & 5 \\ 0 & 0 & 2 & -2 & 5 \end{bmatrix} \begin{matrix} \\ -2 \quad 1 \\ 3 \\ -3 \end{matrix}
$$

$$
\sim \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 0 & -17 & 17 \\ 0 & 0 & 0 & 16 & -16 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{17} \\ \frac{1}{16} \end{matrix}
$$

$$
\sim \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -1 \\ \\ \end{matrix}
$$

$$
\sim \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(REF)}
$$

□

The order in which the 0's were obtained in the preceding example illustrates the general method:

$$
\begin{bmatrix} \bullet & * & * & * & * \\ \textcircled{1} & 0 & \bullet & * & * \\ \textcircled{2} & 0 & \textcircled{4} & \bullet & * \\ \textcircled{3} & 0 & \textcircled{5} & \textcircled{6} & 0 \end{bmatrix}
$$

The $\bullet$'s represent the pivot entries and the circled numbers show the order in which the 0's are obtained. (The other 0's just happen to occur.)

In general, a matrix is row equivalent to more than one matrix in row echelon form (unless the matrix has all zero entries) since one can always multiply a row by a nonzero constant. By adding more stringent conditions to those for row echelon form we get a new form called *reduced* row echelon form. A matrix is row equivalent to one and only one matrix in reduced row echelon form.

Here is a matrix in *reduced* row echelon form:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 7 & -5 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in row echelon form (REF), but it has additional features as well to make it "reduced":

First, the entries above (and below) the pivot entries are all 0's. (The 0's below each pivot entry are already a feature of row echelon form so it is just the requirement of 0's above each pivot entry that is new.)

Second, the pivot entries are all 1's.

> REDUCED ROW ECHELON FORM.
>
> A matrix is in **reduced row echelon form** (abbreviated RREF) if
>
> (a) it is in row echelon form,
>
> (b) each entry above (and below) a pivot entry is 0,
>
> (c) each pivot entry is 1.

In the next section, we will see that having the reduced row echelon form (RREF) of the augmented matrix of a system of equations is convenient for writing down the solution to the system.

1.5.2 Example Find the matrix in reduced row echelon form (RREF) that is row-equivalent to the matrix

$$\begin{bmatrix} 0 & 0 & 6 & 10 & -1 \\ 3 & 1 & -2 & -5 & -3 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix}.$$

Solution This is the matrix from Example 1.5.1. We have already found a matrix in row reduced form (REF) that is row-equivalent to it, so we can just use it here without repeating the steps. To get the *reduced* row echelon form, we use the pivot entries to create 0's above those entries and end by changing all of the pivot entries to 1's:

$$\begin{bmatrix} 0 & 0 & 6 & 10 & -1 \\ 3 & 1 & -2 & -5 & -3 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix} \overset{\text{Ex. 1.5.1}}{\sim} \begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ -10 \quad 5 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 1 & -2 & 0 & -8 \\ 0 & 0 & 6 & 0 & 9 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \\ 1 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 9 & 3 & 0 & 0 & -15 \\ 0 & 0 & 6 & 0 & 9 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{9} \\ \frac{1}{6} \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(RREF)}$$

□

In the following example, "2×3 matrix" refers to a matrix having 2 rows and 3 columns, so of this shape:

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}.$$

1.5.3 Example Write a list of all possible 2×3 matrices in reduced row echelon form (RREF).

Solution We organize the matrices according to how many nonzero rows they have (starting with two):

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

1.6 Writing solutions to systems

Once we have the reduced row echelon form (RREF) of the augmented matrix of a system, we can write down the solution. The next three examples illustrate the procedure for each of the three possibilities: unique solution, infinitely many solutions, and no solution.

1.6.1 Example (Unique solution)

Write the solution to a system given that its augmented matrix is row equivalent to the RREF matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution The augmented matrix corresponds to the system

$$\begin{array}{ccccc} x_1 & & & = & 2 \\ & x_2 & & = & -3 \\ & & x_3 & = & 5 \end{array}$$

so there is a unique solution, namely $x_1 = 2$, $x_2 = -3$, and $x_3 = 5$. The solution set is $\{(2, -3, 5)\}$. □

1.6.2 Example (Infinitely many solutions)

Write the solution to a system given that its augmented matrix is row equivalent to the RREF matrix

$$\left[\begin{array}{cccccc|c} 1 & -5 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & -9 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \end{array}\right]$$

Solution The pivot entries appear in the positions of the variables x_1, x_3, x_5, and x_6. These are called the **lead** variables. The remaining variables, x_2 and x_4, are called the **free** variables. These variables are called free because we can let them be any real numbers. We indicate this by writing

$$x_2 = t, \qquad x_4 = s.$$

Then we solve each of the equations in the system for the lead variable in terms of t and s. Taking the first equation as an example, we have

$$x_1 - 5x_2 + 3x_4 = 4 \quad \Rightarrow \quad x_1 - 5t + 3s = 4 \quad \Rightarrow \quad x_1 = 4 + 5t - 3s.$$

Similarly, we get

$$x_3 = 2 + 9s, \qquad x_5 = 8, \qquad x_6 = -6.$$

Therefore, the solution set is

$$\{(4+5t-3s, t, 2+9s, s, 8, -6) \mid s, t \in \mathbf{R}\}.$$

Since we get a solution for every possible choice of the numbers s and t, there are infinitely many solutions. □

1.6.3 Example (No solution)

Write the solution to a system given that its augmented matrix is row equivalent to the RREF matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Solution The last row of the augmented matrix corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1,$$

that is, $0 = 1$. Since this equation is never satisfied, we conclude that the system has no solution. The solution set is $\emptyset$ (empty set). □

The examples show that we can tell whether a system has a unique solution, infinitely many solutions, or no solution by just looking at the reduced row echelon form (RREF) of its augmented matrix. The criteria refer to the "columns" of the matrix, which are its vertical lists of numbers:

a pivot in every column except the augmented column	⇒	unique solution
no pivot in the augmented column and no pivot in at least one other column	⇒	infinitely many solutions
a pivot in the augmented column	⇒	no solution

We can draw these same conclusions from *any* row echelon form of the augmented matrix (not necessarily reduced).

1 – Exercises

1–1 In each case, sketch the lines to decide whether the system has a unique solution, no solution, or infinitely many solutions. Then solve the system either by using the methods of Section 1.1 or by applying row operations to the augmented matrix of the system.

(a)
$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 - x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 - 2x_2 &= -3 \end{aligned}$$

(c)
$$\begin{aligned} 3x_1 + x_2 &= 2 \\ 6x_1 + 2x_2 &= 4 \end{aligned}$$

1–2 Sketch the three lines and try to decide whether there is a unique solution, no solution, or infinitely many solutions. Then solve the system either by using the methods of Section 1.1 or by applying row operations to the augmented matrix of the system.

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 - x_2 &= 1 \\ 5x_1 + 2x_2 &= 6 \end{aligned}$$

1–3 Apply suitable row operations to the matrix on the left to put it in the form of the matrix on the right. Use type IV row operations, if necessary, to avoid fractions.

(a)
$$\begin{bmatrix} 0 & 1 & 4 \\ 2 & -1 & 6 \\ -3 & 5 & 0 \\ 4 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

(b)
$$\begin{bmatrix} 4 & -2 & 5 & 3 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 9 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 5 & 3 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

1–4 Apply suitable row operations to the matrix on the left to put it in the form of the matrix on the right. Use type IV row operations, if necessary, to avoid fractions until the last step.

$$\begin{bmatrix} 1 & -1 & 4 & 6 & 3 \\ 0 & 2 & -3 & -8 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix} \quad \text{(RREF)}$$

1–5 Find the matrix in reduced row echelon form (RREF) that is row-equivalent to the matrix

$$\begin{bmatrix} 4 & 2 & 1 & -7 & -5 \\ -4 & -2 & 0 & 4 & 7 \\ 8 & 4 & 0 & -8 & -7 \\ 4 & 2 & -2 & 2 & -11 \end{bmatrix}.$$

1–6 In each case, write the solution to a system given that its augmented matrix is row equivalent to the given matrix in reduced row echelon form (RREF).

(a) $$\left[\begin{array}{cccc|c} 1 & 7 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

(b) $$\left[\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 1 & 7 & 4 \\ 0 & 1 & 4 & 0 & 9 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 & 6 & 8 \end{array}\right],$$

(c) $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

1–7 A census of quail in Conecuh National Forest is taken in the fall and it is found that there are a_0 adult birds and j_0 juvenile birds. Studies have shown that for each adult one can expect to find one year later 0.2 adults (due to survival) and 1.6 juveniles (due to reproduction), and for each juvenile one can expect to find one year later 0.4 adults (due to maturation and survival) and 1.4 juveniles (due to reproduction).

(a) Write a system of equations that expresses the next year's adult a_1 and juvenile j_1 populations in terms of the current year's numbers, a_0 and j_0.

(b) If the next year's populations are to be 380 adults and 1600 juveniles, what must the current populations be?