2 Matrix algebra

2.1 Addition and scalar multiplication

Two matrices of the same size are added by adding their corresponding entries. For instance,

\[
\begin{bmatrix}
1 & 2 & 3 \\
-4 & 0 & 9
\end{bmatrix} +
\begin{bmatrix}
2 & 5 & 6 \\
4 & 1 & -3
\end{bmatrix} =
\begin{bmatrix}
3 & 7 & 9 \\
0 & 1 & 6
\end{bmatrix}.
\]

Addition of two matrices that are not of the same size is undefined.

A matrix is multiplied by a scalar (i.e., number) by multiplying each entry of the matrix by the scalar. For instance,

\[
3 \begin{bmatrix}
1 & 2 & 3 \\
-4 & 0 & 9
\end{bmatrix} =
\begin{bmatrix}
3 & 6 & 9 \\
-12 & 0 & 27
\end{bmatrix}.
\]

2.2 Multiplication

One could define multiplication of matrices the same way we defined addition (just multiply corresponding entries). However, we define multiplication a different way—a way that is more relevant for linear algebra.

We multiply two matrices by forming the various dot products between the rows of the first matrix and the columns of the second matrix (the “rows” of a matrix are its horizontal lists of numbers and the “columns” are its vertical lists of numbers). For example,

\[
\begin{bmatrix}
3 & -1 \\
-2 & 4 \\
9 & 0
\end{bmatrix}_{3\times2} \cdot
\begin{bmatrix}
8 & 1 \\
5 & 7
\end{bmatrix}_{2\times2} =
\begin{bmatrix}
19 & -4 \\
4 & 26 \\
72 & 9
\end{bmatrix}_{3\times2}.
\]

First, the product is defined only if the number of columns of the first matrix (here 2) is the same as the number of rows of the second matrix (also 2). Put another way, if the sizes are displayed as indicated \(((\text{# of rows}) \times (\text{# of cols}))\), then the inner numbers (blue) need to match. In this case, the resulting matrix has size given by the outer numbers (here 3 \times 2).

To compute, say, the entry in the 2nd row and 1st column of the resulting matrix, take the dot product of the 2nd row of the first matrix with the 1st column of the second matrix \((-2)(8) + (4)(5) = 4\).

In general, the entry in the \(i\)th row and \(j\)th column of the resulting matrix is the dot product of the \(i\)th row of the first matrix with the \(j\)th column of the second matrix. (For these dot products to make sense, the vectors must have the same number of components and this is the case if and only if the inner numbers (blue) match.)
2.2.1 Example  Compute the following product (if defined):
\[
\begin{bmatrix}
1 & -2 \\
0 & 4
\end{bmatrix}
\begin{bmatrix}
8 & 3 \\
5 & -6 \\
-1 & 7
\end{bmatrix}
\]

Solution  The first matrix has size \(2 \times 2\) and the second matrix has size \(3 \times 2\). Since the inner numbers (blue) do not match, the product is undefined.

2.2.2 Example  Compute the following product (if defined):
\[
\begin{bmatrix}
1 & -2 \\
-3 & 4
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
\]

Solution  The first matrix has size \(2 \times 2\) and the second matrix has size \(2 \times 1\), so the product is defined and the resulting matrix has size given by the outside numbers \(2 \times 1\):
\[
\begin{bmatrix}
1 & -2 \\
-3 & 4
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
= \begin{bmatrix}
(1)(5) + (-2)(6) \\
(-3)(5) + (4)(6)
\end{bmatrix}
= \begin{bmatrix}
-7 \\
9
\end{bmatrix}.
\]

(The middle step can be skipped.)

2.2.3 Example  Compute the following product (if defined):
\[
\begin{bmatrix}
1 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
4 & 1 \\
2 & 0 \\
-1 & 5
\end{bmatrix}
\]

Solution  The first matrix has size \(1 \times 3\) and the second matrix has size \(3 \times 2\), so the product is defined and the resulting matrix has size given by the outside numbers \(1 \times 2\):
\[
\begin{bmatrix}
1 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
4 & 1 \\
2 & 0 \\
-1 & 5
\end{bmatrix}
= \begin{bmatrix}
8 & 11
\end{bmatrix}.
\]

Matrix multiplication allows us to write a system of linear equations as a single matrix equation. For example, the system
\[
\begin{align*}
2x_1 + 3x_2 &= 4 \\
-x_1 - 5x_2 &= 1
\end{align*}
\]
can be written
\[
\begin{bmatrix}
2 & 3 \\
-1 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
4 \\
1
\end{bmatrix} \quad (*)
\]
or, using letters,\[\mathbf{A}\mathbf{x} = \mathbf{b},\]
where
\[
\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.
\]
This can be checked by multiplying the two matrices on the left of (*) to get
\[
\begin{bmatrix} 2x_1 + 3x_2 \\ -x_1 - 5x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.
\]
Saying that these two $2 \times 1$ matrices are equal is the same as saying that their entries are equal, which is what the original system says.

2.2.4 Example Write the following system as a matrix equation:
\[
5x_1 - 3x_2 + x_3 = 4 \\
x_1 + 7x_2 - 9x_3 = 6.
\]

Solution The corresponding matrix equation is
\[
\begin{bmatrix} 5 & -3 & 1 \\ 1 & 7 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.
\]

The identity matrix $\mathbf{I}_{n \times n}$ of size $n \times n$ is given by
\[
\mathbf{I}_{1 \times 1} = [1], \quad \mathbf{I}_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
and so forth.

If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{I}$ is the identity matrix of the same size, then $\mathbf{IA} = \mathbf{A}$ and $\mathbf{AI} = \mathbf{A}$, so $\mathbf{I}$ is a multiplicative identity (it acts like the number 1). For instance, if
\[
\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},
\]
then
\[
\mathbf{IA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \mathbf{A},
\]
as the reader can check. Similarly, $\mathbf{AI} = \mathbf{A}$.
2.3 Algebraic properties

Addition and multiplication of matrices satisfy several of the same properties that addition and multiplication of numbers satisfy. We list these properties here as well as some properties involving scalar multiplication.

Theorem. The following properties hold for any matrices \( A \), \( B \), and \( C \) and any scalars \( \alpha \) and \( \beta \) (when the expressions are defined):

(a) \( A + B = B + A \),
(b) \( (A + B) + C = A + (B + C) \),
(c) \( (AB)C = A(BC) \),
(d) \( A(B + C) = AB + AC \),
(e) \( (A + B)C = AC + BC \),
(f) \( \alpha(A + B) = \alpha A + \alpha B \),
(g) \( (\alpha + \beta)A = \alpha A + \beta A \),
(h) \( (\alpha \beta )A = \alpha (\beta A) \),
(i) \( \alpha(AB) = (\alpha A)B \) and also \( \alpha(AB) = A(\alpha B) \).

Because of these properties, matrices can be regarded as generalized numbers. However, there is one main property of number multiplication that does not hold for matrices in general, namely, the commutative property. In other words, it is possible to have matrices \( A \) and \( B \) for which \( AB \neq BA \), as the following example shows.

2.3.1 Example  
Compute \( AB \) and \( BA \), where

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

and conclude that \( AB \neq BA \).

Solution  
We have

\[
AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

and

\[
BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

so \( AB \neq BA \).
2.4 Matrix transpose

The “transpose” of the matrix

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \]

is the matrix

\[ A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}. \]

In general, the transpose of an \( m \times n \) matrix \( A \) (written \( A^T \)) is the \( n \times m \) matrix obtained by writing the rows of \( A \) as columns.

The transpose of a matrix \( A \) can be visualized as the reflection of \( A \) through the \( 45^\circ \) line starting from the first entry of the matrix and sloping downward to the right.

2.4.1 Example  
Find the transposes of the following matrices:

\[ A = \begin{bmatrix} -1 & 2 \\ 3 & 5 \\ 0 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \\ 8 \end{bmatrix}. \]

Solution  Writing rows as columns, we get

\[ A^T = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 5 & -7 \end{bmatrix}, \quad B^T = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \\ 8 \end{bmatrix}. \]

There are many uses for the transpose of a matrix, but the first we encounter is simply a notational convenience: Instead of writing

\[ B = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \\ 8 \end{bmatrix}, \]

which takes up a lot of vertical space, we will often write \( B = [6, -2, 5, 1, 8]^T \) (commas inserted for clarity), which is more compact and says the same thing.

A matrix \( A \) is symmetric if \( A^T = A \) (i.e., if the transpose is the same as the
original matrix). For instance, the matrix

\[
A = \begin{bmatrix}
1 & 4 & 5 \\
4 & 2 & 6 \\
5 & 6 & 3
\end{bmatrix}
\]

is symmetric.

2–Exercises

2–1 Let

\[
A = \begin{bmatrix}
5 & -1 & 2 \\
3 & 0 & -4
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 1 \\
8 & -6 \\
0 & 3
\end{bmatrix}, \quad C = \begin{bmatrix}
-7 & 4 \\
2 & 0 \\
-9 & 3
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 9 & -3
\end{bmatrix}.
\]

Compute each of the following (or, if the expression is undefined, say so):

(a) \(B + C\)  
(b) \(4A\)  
(c) \(AB\)  
(d) \(BA\)  
(e) \(BD\)  
(f) \(DC\)

2–2 Verify that the equation \(A(B + C) = AB + AC\) holds for the matrices

\[
A = \begin{bmatrix}
-3 & 0 & 2 \\
1 & 5 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
6 \\
-2 \\
-7
\end{bmatrix}, \quad C = \begin{bmatrix}
-5 \\
1 \\
-7
\end{bmatrix}.
\]
An elementary matrix is a matrix obtained by applying to an identity matrix (see Section 2.2) a single elementary row operation (type I, II, or III). For instance, the elementary matrix of size \(3 \times 3\) corresponding to the row operation “add 3 times row one to row two” is

\[
\begin{bmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

since this is what we get when we do the following:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
& & \\
3 & & \\
0 & & \\
\end{bmatrix}
\]

(a) Find the elementary matrices of size \(3 \times 3\) corresponding to each of the following row operations (i) “interchange rows one and two,” (ii) “multiply row three by 5,” (iii) “add \(-2\) times row three to row two.”

(b) Compute the products \(EA\) for each elementary matrix \(E\) found in part (a), where

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}
\]

(c) Use your findings in part (b) to make a conjecture beginning with “If \(E\) is an elementary matrix corresponding to a particular row operation, then for any matrix \(A\) (for which \(EA\) is defined) the matrix \(EA\) is . . . .”

We have seen that the identity matrix \(I\) acts as a multiplicative identity (so it is a generalization of the number 1). For a given matrix \(A\), it sometimes happens that there exists a matrix \(A^{-1}\) such that \(A^{-1}A = I\), so that \(A^{-1}\) acts as a multiplicative inverse of \(A\) (just like with numbers, where \(a^{-1}a = 1\)). Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

and put \(D = ad - bc\). Show that if \(D \neq 0\), then \(A^{-1}A = I\), where

\[
A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

(We need \(D \neq 0\) in order for the fraction \(\frac{1}{D}\) to make sense.)
2–5

(a) Write the following system as a matrix equation, that is, in the form $Ax = b$:

\[3x_1 + 4x_2 = 4\]
\[-x_1 - 2x_2 = 2.\]

(b) Use the formula for $A^{-1}$ found in Exercise 2–4 to solve the system. (Hint: Solve $Ax = b$ for $x$ just like you would solve $ax = b$ for $x$.)

2–6

Find the transposes of the following matrices:

(a) $A = \begin{bmatrix} 6 & 2 & -1 & 5 \\ 4 & 9 & 1 & 0 \end{bmatrix}$,

(b) $B = \begin{bmatrix} 2 & 9 & -3 & 4 & 6 \end{bmatrix}$,

(c) $C = \begin{bmatrix} 1 & 8 & 2 \\ 8 & -7 & 0 \\ 2 & 0 & -3 \end{bmatrix}$.

2–7

Next fall’s adult $a_1$ and juvenile $j_1$ quail populations in Conecuh National Forest depend on this fall’s adult $a_0$ and juvenile $j_0$ populations according to the following equations (see Exercise 1–7):

\[a_1 = 0.2a_0 + 0.4j_0\]
\[j_1 = 1.6a_0 + 1.4j_0.\]

(a) Write this system as a matrix equation, that is, in the form $p_1 = Ap_0$, where $p_i = [a_i, j_i]^T$ (= quail population matrix corresponding to year $i$).

(b) Express the quail populations (adult and juvenile) two years from this fall in terms of this fall’s populations by finding a matrix $B$ such that $p_2 = Bp_0$. (Hint: Use part (a) and the fact that $p_2 = Ap_1$.)

(c) Find the quail populations two years from this fall assuming that this fall there are 300 adults and 800 juveniles. (Hint: Use part (b).)