5. Linear independence

5.1. Introduction

Let \( \mathbf{x}_1, \mathbf{x}_2, \) and \( \mathbf{x}_3 \) be three vectors in \( \mathbb{R}^n \). There is always one way to get a linear combination of these vectors to equal zero, namely,

\[
0\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 = \mathbf{0}.
\]

But suppose that there’s another way. For instance,

\[
2\mathbf{x}_1 + 5\mathbf{x}_2 + 4\mathbf{x}_3 = \mathbf{0}.
\]

In this case, we say that the vectors are “linearly dependent.” The reason for the terminology is that we can solve for one of the vectors in terms of the others, say,

\[
\mathbf{x}_1 = -\frac{5}{2}\mathbf{x}_2 - 2\mathbf{x}_3.
\]

So \( \mathbf{x}_1 \) “depends” on the other two vectors.
5.2. Definition and examples

**Linear Dependence/Independence.**

We say that vectors $x_1, x_2, \ldots, x_s$ in $\mathbb{R}^n$ are **linearly dependent** if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_s$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_s x_s = 0.$$

We say that the vectors are **linearly independent** if they are not linearly dependent, that is, if

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_s x_s = 0 \implies \alpha_i = 0 \text{ for all } i.$$

In other words, the vectors $x_1, x_2, \ldots, x_s$ are linearly dependent if there is a way to get a linear combination of them to equal 0 without making all of the scalar factors 0, and, on the other hand, they are linearly independent if the only way to get a linear combination of them to equal 0 is by making all of the scalar factors 0.

We say that the set $\{x_1, x_2, \ldots, x_2\}$ is linearly dependent if the vectors $x_1, x_2, \ldots, x_s$ are linearly dependent (and similarly for linearly independent).

**5.2.1 Example**  Determine whether the following vectors in $\mathbb{R}^2$ are linearly dependent or linearly independent:

$$x_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$
Solution  Suppose we have a linear combination of the vectors equal to 0:

\[ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \]

\[ \alpha_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} -\alpha_1 + 5\alpha_2 + \alpha_3 \\ 3\alpha_1 + 6\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

Equating components we get a system with augmented matrix

\[
\begin{bmatrix}
-1 & 5 & 1 & 0 \\
3 & 6 & 4 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
-1 & 5 & 1 & 0 \\
0 & 21 & 7 & 0
\end{bmatrix}.
\]

Since \( \alpha_3 \) is free, we can choose it to be anything. In particular, we can choose it to be nonzero. Therefore, the vectors are linearly dependent.

5.2.2 Example  Determine whether the following vectors in \( \mathbb{R}^3 \) are linearly dependent or linearly independent:

\[ x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \]
Solution  Suppose we have a linear combination of the vectors equal to 0:

\[ \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 = 0 \]

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix}
+ \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
\begin{bmatrix}
-2 \\
0 \\
1 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Equating components we get a system with augmented matrix

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 5 & -2 & 0 \\
0 & 6 & -2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 5 & -2 & 0 \\
0 & 5 & -2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Since there is a pivot in every column except for the augmented column, there is a unique solution, namely, \( \alpha_1 = 0, \alpha_2 = 0, \) and \( \alpha_3 = 0. \)

The computation shows that the only way to get a linear combination of the vectors to equal 0 is by making all of the scalar factors 0. Therefore, the vectors are linearly independent.

The solutions to these last two examples show that the question of whether some given vectors are linearly independent can be answered just by looking at a row-reduced form of...
the matrix obtained by writing the vectors side by side. The following theorem uses a new term: A matrix has **full rank** if a row-reduced form of the matrix has a pivot in every column.

**Theorem.** Let \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s \) be vectors in \( \mathbb{R}^n \) and let \( \mathbf{A} \) be the matrix formed by writing these vectors side by side:

\[
\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_s \end{bmatrix}.
\]

The vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s \) are linearly independent if and only if \( \mathbf{A} \) has full rank.

5.2.3 **Example**  Use the last theorem to determine whether the vectors \([1, 3, -1, 0]^T\), \([4, 9, -2, 1]^T\), and \([2, 3, 0, 1]^T\) are linearly independent.

**Solution**  We have

\[
\begin{bmatrix}
1 & 4 & 2 \\
3 & 9 & 3 \\
-1 & -2 & 0 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
-3 \\
1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 4 & 2 \\
0 & -3 & -3 \\
0 & 2 & 2 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 4 & 2 \\
0 & -3 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Since the matrix does not have full rank, the vectors are *not* linearly independent.
5.3. Facts about linear dependence/independence

The next theorem says that if a vector is written as a linear combination of linearly independent vectors, then the scaling factors are uniquely determined.

**Theorem.** Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ be linearly independent vectors in $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ and suppose that

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_s \mathbf{x}_s$$

and also

$$\mathbf{x} = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \cdots + \beta_s \mathbf{x}_s,$$

with $\alpha_i, \beta_i \in \mathbb{R}$. Then $\alpha_i = \beta_i$ for each $i$.

**Proof.** We have

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_s \mathbf{x}_s = \mathbf{x} = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \cdots + \beta_s \mathbf{x}_s,$$

so moving all terms to the left-hand side and collecting like terms, we get

$$(\alpha_1 - \beta_1) \mathbf{x}_1 + (\alpha_2 - \beta_2) \mathbf{x}_2 + \cdots + (\alpha_s - \beta_s) \mathbf{x}_s = \mathbf{0}.$$ 

Since the vectors are linearly independent, the only way to get a linear combination of them to equal $\mathbf{0}$ is by making all of the scalar factors 0. We conclude that $\alpha_i - \beta_i = 0$ for all $i$, that is, $\alpha_i = \beta_i$ for all $i$. \qed
Theorem. The vectors \( x_1, x_2, \ldots, x_s \) (\( s \geq 2 \)) in \( \mathbb{R}^n \) are linearly dependent if and only if one of the vectors can be written as a linear combination of the others.

Due to the “if and only if” phrase, the theorem is saying two things: (1) if the vectors are linearly dependent, then one can be written as a linear combination of the others, and (2) if one vector can be written as a linear combination of the others, then the vectors are linearly dependent.

We omit the proof, but the following example illustrates the main ideas:

\textbf{5.3.1 Example}

(a) Let \( x_1, x_2, x_3, x_4 \) be vectors in \( \mathbb{R}^n \) and suppose that

\[ 3x_1 - 5x_2 + 0x_3 + 7x_4 = 0, \]

which shows that the vectors are linear dependent. Actually write one as a linear combination of the others.

(b) Let \( x_1, x_2, x_3, x_4, x_5 \) be vectors in \( \mathbb{R}^n \) and suppose that one of the vectors can be written as a linear combination of the others, say,

\[ x_4 = 3x_1 + 0x_2 + (-2)x_3 + 0x_5. \]

Show that the vectors are linearly dependent.

\textit{Solution} \quad (a) We can solve the equation for any of the vectors with nonzero scalar factor,
say $x_1$:

$$3x_1 = 5x_2 + 0x_3 - 7x_4$$
$$x_1 = \frac{5}{3}x_2 + 0x_3 - \frac{7}{3}x_4.$$

It is the last step of dividing both sides by the scalar factor 3 that uses the fact that it is nonzero.

(We could have solved for either $x_2$ or $x_4$ as well, but not $x_3$.)

(b) Moving all terms to the left, we get

$$-3x_1 + 0x_2 + 2x_3 + 1x_4 + 0x_5 = 0.$$

This is a linear combination of the vectors equaling 0 with not all scalar factors equal to 0 (the vector $x_4$ has scalar factor 1). Therefore, the vectors are linearly dependent.

A linearly dependent list of vectors has a redundancy in the sense that one of the vectors can be removed without affecting the span. The next example illustrates this.

5.3.2 Example Let $x_1$, $x_2$, and $x_3$ be vectors in $\mathbb{R}^n$ and let $S = \text{Span}\{x_1, x_2, x_3\}$. Show that if the vectors $x_1$, $x_2$, and $x_3$ are linearly dependent, then $S$ is the span of two of these vectors.

Solution Assume that the vectors $x_1$, $x_2$, and $x_3$ are linearly dependent. By the previous theorem, one of the vectors is a linear combination of the others. By renumbering the vectors, if necessary, we may assume that this vector is $x_3$, so that $x_3 = \alpha_1 x_1 + \alpha_2 x_2$ for some scalars $\alpha_1$ and $\alpha_2$. 
We claim that $S = \text{Span}\{x_1, x_2\}$. To show that these two sets are equal we show that each is a subset of the other. Let $x$ be a vector in $S$. Then $x = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$ for some scalars $\beta_1$, $\beta_2$, and $\beta_3$. Therefore,

$$x = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

$$= \beta_1 x_1 + \beta_2 x_2 + \beta_3(\alpha_1 x_1 + \alpha_2 x_2)$$

$$= (\beta_1 + \beta_3 \alpha_1)x_1 + (\beta_2 + \beta_3 \alpha_2)x_2$$

and this shows that $x$ is in $\text{Span}\{x_1, x_2\}$. Thus, $S \subseteq \text{Span}\{x_1, x_2\}$.

For the other inclusion, we note that $x_1$ and $x_2$ are in $S$ (see Exercise 4–2) and $S$ is a subspace of $\mathbb{R}^n$ (see Section 4.4) so every linear combination of these vectors must also lie in $S$ by the closure properties of subspace. Therefore, $\text{Span}\{x_1, x_2\} \subseteq S$.

This establishes the claim that $S = \text{Span}\{x_1, x_2\}$.

**Theorem.** The vectors $x_1, x_2, \ldots, x_s$ are linearly dependent in either of the following cases:

(a) One vector is a multiple of another,

(b) One of the vectors equals 0.

**Proof.** (a) Assume that one vector is a multiple of the another. By renumbering the vectors if necessary, we may assume that $x_1 = \alpha x_2$ for some $\alpha \in \mathbb{R}$. Then

$$x_1 = \alpha x_2 + 0x_3 + \cdots + 0x_s.$$

Since one of the vectors is a linear combination of the others, the vectors are linearly dependent (see last theorem).
(b) Assume that one of the vectors equals 0. By renumbering the vectors if necessary, we may assume that this vector is $x_1$. Then

\[(1)x_1 + 0x_2 + \cdots + 0x_n = 0,\]

and, since not all of the scalar factors equal zero (the first is not zero), we conclude that the vectors are linearly dependent.

5.3.3 Example In each case, use inspection to tell whether the vectors are linearly independent:

(a) \[
\begin{bmatrix}
-1 \\
3 \\
2
\end{bmatrix}, \begin{bmatrix}
0 \\
4 \\
9
\end{bmatrix}, \begin{bmatrix}
2 \\
-6 \\
-4
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 \\
-4 \\
7
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
-8
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

Solution (a) The third vector is $-2$ times the first vector, so the vectors are linearly dependent by the last theorem.

(b) Neither vector is a linear combination of the other vector (meaning multiple of the other vector in this case), so the vectors are linearly independent.
(c) The second vector is \( \mathbf{0} \), so the vectors are linearly dependent by the last theorem.

(d) The matrix having these three vectors as columns has full rank, so the vectors are linearly independent.
5–Exercises

5–1 In each case, use only the definitions to determine whether the vectors are linearly dependent or linearly independent. If the vectors are linearly dependent, then actually write one as a linear combination of the others (see Example 5.3.1 and the theorem preceding it).

(a) \( \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \)

(b) \( \mathbf{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -6 \\ 7 \\ 9 \end{bmatrix} \)

5–2 Use the theorem before Example 5.2.3 to determine whether the vectors \([2, -2, -6, -2]^T, \quad [-1, 5, 3, -3]^T, \quad [3, -5, -4, -1]^T\) are linearly independent.

5–3 Let \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \) and \( \mathbf{x}_4 \) be linearly independent vectors in \( \mathbb{R}^n. \) Show that \( \mathbf{x}_1, \mathbf{x}_2, \) and \( \mathbf{x}_3 \) are linearly independent.
5–4 In each case, use inspection to tell whether the vectors are linearly independent:

(a) \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
2 & 3 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
4 & 5 \\
6 & 6
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & 2 \\
-2 & -4 \\
4 & 8 \\
3 & 7
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 1
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
5 & 4 & 0 \\
2 & 3 & 0 \\
-3 & 9 & 0
\end{bmatrix}
\]

5–5 Let \( \mathbf{x}_1, \mathbf{x}_2, \) and \( \mathbf{x}_3 \) be linearly independent vectors in \( \mathbb{R}^n \). Show that if \( \mathbf{x}_4 \) is a vector in \( \mathbb{R}^n \) that is not in \( \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \), then \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \) and \( \mathbf{x}_4 \) are linearly independent.