11 Invertibility

11.1 Introduction

If $f(x) = x^3 + 1$, then the inverse of f is given by $f^{-1}(x) = \sqrt[3]{x-1}$. The inverse function f^{-1} undoes what f does to an input:

$$f^{-1}(f(x)) = f^{-1}(x^3+1) = \sqrt[3]{(x^3+1)-1} = x$$

(following f by f^{-1} gets one back to the original input). It is also true that $f(f^{-1}(x)) = x$ so that f undoes what f^{-1} does as well.

The inverse of a function f is useful for solving $f(x) = b$ for x (i.e., finding the input that produces the output b). We have

$$\begin{aligned} f(x) &= b \\ f^{-1}(f(x)) &= f^{-1}(b) \\ x &= f^{-1}(b). \end{aligned}$$

In this section, we study the analogous notion for a linear function $L : V \to V'$.

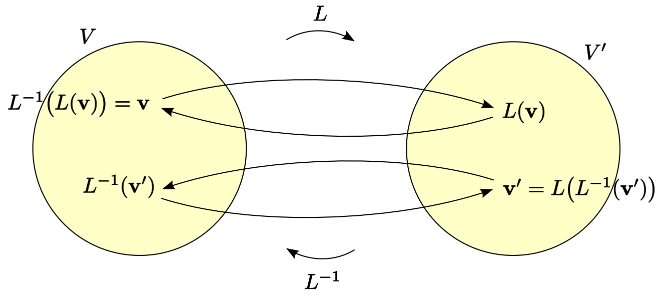
11.2 Inverse function

Inverse function.

Let $L : V \to V'$ be a linear function. An **inverse** of L is a linear function $L^{-1} : V' \to V$ such that

$$L^{-1}(L(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad L(L^{-1}(\mathbf{v}')) = \mathbf{v}'$$

for all $\mathbf{v} \in V$ and $\mathbf{v}' \in V'$.



The function L is **invertible** if it has an inverse.

An invertible function has only one inverse. (If L^{-1} and $\hat{L}^{-1}$ are both inverses of L, then for all $\mathbf{v}' \in V'$, we have $\hat{L}^{-1}(\mathbf{v}') = \hat{L}^{-1}\big(L(L^{-1}(\mathbf{v}'))\big) = L^{-1}(\mathbf{v}')$, so that $\hat{L}^{-1} = L^{-1}$.)

11.2.1 Example Let $L : \mathbf{R}^2 \to \mathbf{P}_2$ be the linear function given by

$$L(\mathbf{x}) = (x_1 + 2x_2)x + (x_1 + 3x_2).$$

(a) Show that L is invertible with inverse $L^{-1} : \mathbf{P}_2 \to \mathbf{R}^2$ given by $L^{-1}(ax + b) = [3a - 2b, -a + b]^T$.

(b) Find $\mathbf{x}$ such that $L(\mathbf{x}) = 5x + 8$.

Solution

(a) For all $\mathbf{x}$ in $\mathbf{R}^2$, we have

$$L^{-1}\big(L(\mathbf{x})\big) = L^{-1}\big((x_1 + 2x_2)x + (x_1 + 3x_2)\big) = \begin{bmatrix} 3(x_1 + 2x_2) - 2(x_1 + 3x_2) \\ -(x_1 + 2x_2) + (x_1 + 3x_2) \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x},$$

and for all $ax + b \in \mathbf{P}_2$, we have

$$L\big(L^{-1}(ax + b)\big) = L\left(\begin{bmatrix} 3a - 2b \\ -a + b \end{bmatrix}\right) = \big((3a - 2b) + 2(-a + b)\big)x + \big((3a - 2b) + 3(-a + b)\big)$$
$$= ax + b.$$

(b) As was demonstrated in the introduction, the answer is

$$\mathbf{x} = L^{-1}(5x + 8) = \begin{bmatrix} 3(5) - 2(8) \\ -(5) + (8) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(Check: $L\big([-1, 3]^T\big) = \big((-1) + 2(3)\big)x + \big((-1) + 3(3)\big) = 5x + 8$.)

□

THEOREM. *Let $L : V \to V'$ be a linear function. L is invertible if and only if both*

(i) $\ker L = \{\mathbf{0}\}$ *and*

(ii) $\operatorname{im} L = V'$.

Proof.

($\Rightarrow$) Assume that L is invertible with inverse $L^{-1} : V' \to V$.

(i) Let $\mathbf{v} \in \ker L$. Then

$$\begin{aligned} \mathbf{v} &= L^{-1}(L(\mathbf{v})) && \text{definition of inverse} \\ &= L^{-1}(\mathbf{0}) && \mathbf{v} \in \ker L \\ &= \mathbf{0} && \text{property of linear function} \end{aligned}$$

so $\mathbf{v} \in \{\mathbf{0}\}$. This shows that $\ker L \subseteq \{\mathbf{0}\}$. The other inclusion holds since $\ker L$ is a subspace of V.

(ii) If $\mathbf{v}' \in V'$, then $\mathbf{v}' = L\big(L^{-1}(\mathbf{v}')\big) \in \operatorname{im} L$, so $V' \subseteq \operatorname{im} L$. The other inclusion holds by how L is defined.

($\Leftarrow$) Assume that L satisfies (i) and (ii). Let $\mathbf{v}' \in V'$. Since $\operatorname{im} L = V'$, we know that $\mathbf{v}' = L(\mathbf{v})$ for some $\mathbf{v} \in V$. If also $\mathbf{v}' = L(\mathbf{w})$, then $L(\mathbf{v} - \mathbf{w}) = L(\mathbf{v}) - L(\mathbf{w}) = \mathbf{v}' - \mathbf{v}' = \mathbf{0}$ so that $\mathbf{v} - \mathbf{w} \in \ker L = \{\mathbf{0}\}$, that is $\mathbf{v} - \mathbf{w} = \mathbf{0}$, so that $\mathbf{v} = \mathbf{w}$. Therefore, $\mathbf{v}$ is the unique vector in V such that $L(\mathbf{v}) = \mathbf{v}'$. Define $L^{-1} : V' \to V$ by letting $L^{-1}(\mathbf{v}')$ be the unique vector $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{v}'$. Then L has inverse L^{-1}. □

11.2.2 Example Let $L : \mathbf{D_R} \to \mathbf{F_R}$ be the derivative function given by

$$L(f) = f'.$$

Is L invertible? Explain.

Solution The function L is not invertible since the constant function 1 is a nonzero function in the kernel of L, so that $\ker L \neq \{\mathbf{0}\}$. □

The next theorem says that if L is a linear operator on a finite dimensional vector space, then it is invertible if *either* (i) or (ii) in the last theorem holds.

THEOREM. *Let $L : V \to V$ be a linear operator on V and assume that V is finite-dimensional. L is invertible if* either

(i) $\ker L = \{\mathbf{0}\}$ *or*

(ii) $\operatorname{im} L = V$.

Proof. Let n be the dimension of V. By the last theorem of Section 8.13,

$$\dim \operatorname{im} L + \dim \ker L = n.$$

If $\ker L = \{\mathbf{0}\}$, then $\dim \ker L = 0$, so this equation forces $\dim \operatorname{im} L$ to be n, which in turn implies that $\operatorname{im} L = V'$ ($\operatorname{im} L$ is a subspace of V and since they both have the same dimension, they must be equal). By the last theorem, L is invertible.

On the other hand, if $\operatorname{im} L = V$, then $\dim \operatorname{im} L = n$, so the equation forces $\dim \ker L = 0$, which in turn implies that $\ker L = \{\mathbf{0}\}$. Again, the last theorem says that L is invertible. □

11.2.3 Example Decide whether each of the following linear functions is invertible. If the function is invertible, find its inverse.

(a) $L : \mathbf{R}^2 \to \mathbf{R}^2$ described by "90° counterclockwise rotation."

(b) $L : \mathbf{R}^2 \to \mathbf{R}^2$ described by "projection onto the x_1-axis."

Solution

(a) The only vector that L sends to $\mathbf{0}$ is $\mathbf{0}$, so $\ker L = \{\mathbf{0}\}$. By the preceding theorem, L is invertible. In fact, L^{-1} is "90° clockwise rotation."

(b) The vector $[0, 1]^T$ is a nonzero vector in $\ker L$, so $\ker L \neq \{\mathbf{0}\}$ and L is not invertible by the first theorem above.

□

11.3 Inverse matrix

Inverse matrix.

Let $\mathbf{A}$ be an $n \times n$ matrix. An **inverse** of $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \text{and} \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The matrix $\mathbf{A}$ is **invertible** (or **nonsingular**) if it has an inverse.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$$

is invertible with inverse

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix},$$

since

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

and

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

Actually, if the $n \times n$ matrix $\mathbf{A}^{-1}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, then it will automatically satisfy $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ as well. Similarly, if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, then automatically $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Therefore, in order to verify that one matrix is an inverse of another, it is only necessary to compute one product.

An invertible matrix has only one inverse. (If $\mathbf{A}^{-1}$ and $\hat{\mathbf{A}}^{-1}$ are both inverses of $\mathbf{A}$, then $\hat{\mathbf{A}}^{-1} = \hat{\mathbf{A}}^{-1}\mathbf{I} = \hat{\mathbf{A}}^{-1}\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}\mathbf{A}^{-1} = \mathbf{A}^{-1}$.)

Suppose that a system of equations has corresponding matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. If $\mathbf{A}$ is invertible, then this equation can be solved as follows:

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{I}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

11.3.1 Example Use an inverse matrix to solve the system

$$\begin{aligned} x_1 - x_2 &= 3 \\ -2x_1 + 4x_2 &= 6. \end{aligned}$$

Solution The corresponding matrix equation is $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Using the inverse of $\mathbf{A}$ given above, we get

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix},$$

so $x_1 = 9$ and $x_2 = 6$. □

11.4 Finding an inverse matrix

Let us try to find the inverse (if it exists) of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We seek a matrix

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, that is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the entries, we get the two systems

$$\begin{aligned} a + 2c &= 1 \\ 3a + 4c &= 0 \end{aligned} \qquad\qquad \begin{aligned} b + 2d &= 0 \\ 3b + 4d &= 1 \end{aligned}$$

which have the corresponding augmented matrices

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right].$$

We can solve both systems simultaneously by applying row operations to a doubly augmented matrix:

$$\begin{aligned} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]{-3} \;\; &\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]{1} \\ &\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]{-\tfrac{1}{2}} \\ &\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]. \end{aligned}$$

Therefore, $a = -2$, $c = \frac{3}{2}$, $b = 1$, and $d = -\frac{1}{2}$. We conclude that

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix},$$

which can be checked by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

This procedure for finding the inverse of a matrix generalizes to a square matrix of any size:

> FINDING AN INVERSE MATRIX.
>
> Let $\mathbf{A}$ be an $n \times n$ matrix. The inverse of $\mathbf{A}$ (if it exists) can be found by applying row operations to the augmented matrix $[\,\mathbf{A} \,|\, \mathbf{I}\,]$:
> $$[\,\mathbf{A} \,|\, \mathbf{I}\,] \sim \cdots \sim [\,\mathbf{I} \,|\, \mathbf{A}^{-1}\,].$$
> If $\mathbf{A}$ is not row equivalent to $\mathbf{I}$, then $\mathbf{A}^{-1}$ does not exist.

11.4.1 Example Find the inverse (if it exists) of the following matrix :

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -2 & -6 & -1 \\ -3 & -8 & -3 \end{bmatrix}.$$

Solution Using the theorem, we have

$$[\,\mathbf{A}\,|\,\mathbf{I}\,] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ -2 & -6 & -1 & 0 & 1 & 0 \\ -3 & -8 & -3 & 0 & 0 & 1 \end{array}\right] \begin{matrix} 2 \quad 3 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 2 & 1 & 0 \\ 0 & 1 & 6 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & 0 & 1 \\ 0 & 0 & 5 & 2 & 1 & 0 \end{array}\right] \begin{matrix} \quad 5 \\ 5 \\ -6 \quad -3 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 5 & 15 & 0 & -1 & -3 & 0 \\ 0 & 5 & 0 & 3 & -6 & 5 \\ 0 & 0 & 5 & 2 & 1 & 0 \end{array}\right] \begin{matrix} \\ -3 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & -10 & 15 & -15 \\ 0 & 5 & 0 & 3 & -6 & 5 \\ 0 & 0 & 5 & 2 & 1 & 0 \end{array}\right] \begin{matrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & -3 \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{6}{5} & 1 \\ 0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} & 0 \end{array}\right].$$

Therefore,

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 3 & -3 \\ \frac{3}{5} & -\frac{6}{5} & 1 \\ \frac{2}{5} & \frac{1}{5} & 0 \end{bmatrix},$$

which can be checked by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. □

Here is a formula for the inverse of an (invertible) 2×2 matrix:

Theorem. *Let* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and put* $D = ad - bc$. *If* $D \neq 0$ *then* $\mathbf{A}$ *is invertible and*

$$\mathbf{A}^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. We need only check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. We have

$$\begin{aligned} \mathbf{A}\mathbf{A}^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{D} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{D} \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}, \end{aligned}$$

as desired. □

For instance,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(1)(4) - (2)(3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

in agreement with what we found earlier.

11.5 Theorems about inverses

Theorem. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. The following are equivalent*:

(i) $\mathbf{A}^{-1}$ *exists,*

(ii) $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a unique solution for each* $\mathbf{b} \in \mathbf{R}^n$*,*

(iii) $\mathbf{A}$ *has full rank,*

(iv) $\mathbf{A} \sim \mathbf{I}$.

Proof. (i ⇒ ii) Assuming (i), for each $\mathbf{b} \in \mathbf{R}^n$ we can solve the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ to get the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, so (ii) follows.

(ii ⇒ iii) Assume (ii) holds. Then $\mathbf{A}\mathbf{x} = \mathbf{0}$ has a unique solution. Therefore, no free variable will arise when we put the corresponding augmented matrix $[\,\mathbf{A} \,|\, \mathbf{0}\,]$ in REF. This implies that $\mathbf{A}$ has full rank, so (iii) follows.

(iii ⇒ iv) Assuming (iii), we see that the RREF of $\mathbf{A}$ is $\mathbf{I}$, so (iv) follows.

(iv ⇒ i) Assume (iv) holds. Applying row operations to $[\,\mathbf{A} \,|\, \mathbf{I}\,]$ will transform the left half into $\mathbf{I}$ and, as was shown in Section 11.4, this process will transform the right half into $\mathbf{A}^{-1}$:

$$[\,\mathbf{A} \,|\, \mathbf{I}\,] \sim \cdots \sim [\,\mathbf{I} \,|\, \mathbf{A}^{-1}\,].$$

Therefore (i) follows. □

Theorem. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *invertible matrices. The product* $\mathbf{AB}$ *is invertible and*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

Proof. We have

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I},$$

so $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ as claimed. □

Theorem. *Let* $L : \mathbf{R}^n \to \mathbf{R}^n$ *be a linear function and let* $\mathbf{A}$ *be the matrix of* L*. If* $\mathbf{A}$ *is invertible, then* L *is invertible and* L^{-1} *has matrix* $\mathbf{A}^{-1}$*.*

Proof. Assume that $\mathbf{A}$ is invertible. Define $M : \mathbf{R}^n \to \mathbf{R}^n$ by $M(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x}$. Then for every $\mathbf{x} \in \mathbf{R}^n$ we have

$$M(L(\mathbf{x})) = M(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = (\mathbf{A}^{-1}\mathbf{A})(\mathbf{x}) = \mathbf{I}\mathbf{x} = \mathbf{x}$$

and similarly $L(M(\mathbf{x})) = \mathbf{x}$. Therefore, $M = L^{-1}$ and, by the way M was defined, it follows that L^{-1} has matrix $\mathbf{A}^{-1}$. □

11.5.1 Example Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be the function given by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

Find a formula for L^{-1} (if this inverse exists).

Solution The matrix of L is

$$\mathbf{A} = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix},$$

which has inverse

$$\mathbf{A}^{-1} = \frac{1}{(1)(3) - (2)(1)} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

By the theorem, L^{-1} exists and its matrix is $\mathbf{A}^{-1}$. So, for every $\mathbf{x} \in \mathbf{R}^2$,

$$L^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ -x_1 + x_2 \end{bmatrix}.$$

□

11 – Exercises

11–1 Let $L : \mathbf{P}_4 \to \mathbf{M}_{2\times 2}$ be the linear function given by

$$L(ax^3 + bx^2 + cx + d) = \begin{bmatrix} 2a - b & a \\ d & -c + 3d \end{bmatrix}.$$

(a) Show that L is invertible with inverse $L^{-1} : \mathbf{M}_{2\times 2} \to \mathbf{P}_4$ given by

$$L^{-1}\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \beta x^3 + (-\alpha + 2\beta)x^2 + (3\gamma - \delta)x + \gamma.$$

(b) Find p such that $L(p) = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

11–2 Decide whether each of the following linear functions is invertible. If the function is invertible, find its inverse.

(a) $L : \mathbf{F_R} \to \mathbf{R}$ by $L(f) = f(0)$.

(b) $L : \mathbf{R}^2 \to \mathbf{R}^2$ described by "reflection across the line $x_2 = x_1$."

11–3 Find the inverse (if it exists) of the following matrix :

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}.$$

11–4 Use an inverse matrix to solve the system

$$\begin{aligned} x_1 \qquad + x_3 &= 5 \\ -x_1 + x_2 + x_3 &= 2 \\ -x_1 - 2x_2 - 3x_3 &= -3 \end{aligned}$$

HINT: Example 11.3.1 and Exercise 11–3.

11–5

(a) Use the formula for the inverse of a 2×2 matrix given in the last theorem of Section 11.4 to find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 9 & 5 \end{bmatrix}.$$

(b) Use part (a) to find a 2×2 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$ where

$$\mathbf{B} = \begin{bmatrix} 3 & -1 \\ -6 & 7 \end{bmatrix}.$$

11–6 Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear function given by

$$L(\mathbf{x}) = \begin{bmatrix} 6x_1 + 2x_2 \\ 4x_1 + x_2 \end{bmatrix}.$$

Find a formula for L^{-1} (if this inverse exists).