7 Dimension

7.1 Introduction

The reader has often heard the plane referred to as being "two dimensional". Intuitively, what this means is that, in the plane, one can move in two directions: side to side or up and down. By combining movements in these directions, one can move anywhere in the plane.

The notion of basis allows us to make this terminology precise: The plane $\mathbf{R}^2$ has dimension two because it has a basis consisting of two vectors, namely, $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$. The first vector and its multiples allow side-to-side movement; the second vector and its multiples allow up-and-down movement. Combinations of these movements correspond to linear combinations of these two vectors and, since these vectors span the plane, any position in the plane can be reached by using these two movements.

We have seen that $\{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = [3, 1]^T$ and $\mathbf{b}_2 = [1, 2]^T$ is also a basis for $\mathbf{R}^2$. The vector $\mathbf{b}_1$ allows for movement in roughly a side-to-side direction and the vector $\mathbf{b}_2$ allows for movement in roughly an up-and-down direction. Again, combinations of these movements allow one to reach any position in the plane.

It is no accident that both bases of the plane, $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$, consist of two vectors. It is a fact that every basis of the plane must consist of two vectors. This allows one to define the dimension of $\mathbf{R}^2$ without referring to a particular basis: the dimension of $\mathbf{R}^2$ is the number of vectors in any (and hence every) basis, namely two.

In this section, we generalize this discussion and define the dimension of any subspace S of $\mathbf{R}^n$ (including $\mathbf{R}^n$ itself) to be the number of vectors in any basis of S. The crucial first step in this definition is showing that any two bases of S must have the same number of vectors.

7.2 Definition and examples

The following theorem is the key step in showing that any two bases of a subspace have the same number of vectors.

THEOREM. *Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s$ be vectors in $\mathbf{R}^n$ and let S be their span. If $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$ are vectors in S and $t > s$, then $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$ are linearly dependent.*

Proof. We prove only the special case $s = 2$, $t = 3$, since this will illustrate the main ideas in the general proof. So we are assuming that $\mathbf{y}_1, \mathbf{y}_2$ span S and we are trying to show that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly dependent.

Since the vectors $\mathbf{y}_1, \mathbf{y}_2$ span S, we can write each of the vectors $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ as a linear combination of $\mathbf{y}_1$ and $\mathbf{y}_2$:

$$\begin{aligned}
\mathbf{z}_1 &= \beta_{11}\mathbf{y}_1 + \beta_{21}\mathbf{y}_2, \\
\mathbf{z}_2 &= \beta_{12}\mathbf{y}_1 + \beta_{22}\mathbf{y}_2, \\
\mathbf{z}_3 &= \beta_{13}\mathbf{y}_1 + \beta_{23}\mathbf{y}_2.
\end{aligned}$$

The system of linear equations

$$\begin{aligned}
\beta_{11}x_1 + \beta_{12}x_2 + \beta_{13}x_3 &= 0 \\
\beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3 &= 0
\end{aligned}$$

has infinitely many solutions (has at least one solution due to the zeros on the right, and fewer equations than unknowns so a free variable will occur), so we can choose a solution $\alpha_1, \alpha_2, \alpha_3$ not all zero. Then

$$\begin{aligned}
\alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \alpha_3\mathbf{z}_3 &= \alpha_1(\beta_{11}\mathbf{y}_1 + \beta_{21}\mathbf{y}_2) + \alpha_2(\beta_{12}\mathbf{y}_1 + \beta_{22}\mathbf{y}_2) + \alpha_3(\beta_{13}\mathbf{y}_1 + \beta_{23}\mathbf{y}_2) \\
&= (\beta_{11}\alpha_1 + \beta_{12}\alpha_2 + \beta_{13}\alpha_3)\mathbf{y}_1 + (\beta_{21}\alpha_1 + \beta_{22}\alpha_2 + \beta_{23}\alpha_3)\mathbf{y}_2 \\
&= 0\mathbf{y}_1 + 0\mathbf{y}_2 \\
&= \mathbf{0}.
\end{aligned}$$

Since $\alpha_1, \alpha_2, \alpha_3$ are not all zero, it follows that $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ are linearly dependent. □

Put another way, the theorem says that if a subspace S of $\mathbf{R}^n$ is spanned by s vectors then any list of linearly independent vectors in S must consist of at most s vectors.

THEOREM. *If $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\}$ and $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_t\}$ are both bases for a subspace S of $\mathbf{R}^n$, then $s = t$.*

Proof. Assume that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\}$ and $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_t\}$ are both bases for the subspace S of $\mathbf{R}^n$. Since the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s$ span S, and the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_t$ are linearly independent, the preceding theorem implies that $t \leq s$. Reversing the roles of the $\mathbf{b}_i$'s and the $\mathbf{c}_i$'s, we also get $s \leq t$. Therefore, $s = t$ as desired. □

DIMENSION.

Let S be a subspace of $\mathbf{R}^n$. If S has a basis consisting of s vectors, we say that S has **dimension** s and we write $\dim S = s$. By convention, the subspace $\{\mathbf{0}\}$ has the empty set $\emptyset$ as basis and therefore dimension 0.

We emphasize that this definition makes sense only in view of the preceding theorem, which says that any two bases for S must consist of the same number of vectors. For, if this were not the case, one person might find a basis consisting of two vectors and say that the dimension of S is two, while another person might find a basis consisting of three vectors and say that the dimension of S is three.

7.2.1 Example Find the dimension of $\mathbf{R}^3$.

Solution Since the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$, we have $\dim \mathbf{R}^3 = 3$. □

Similarly, the dimension of $\mathbf{R}^n$ is n for any n.

7.2.2 Example Find the dimension of $S = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution In Example 6.2.3 it was shown that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for S. Therefore, $\dim S = 2$. □

7.2.3 Example Let S be a subspace of $\mathbf{R}^n$ and assume that $\dim S = t$. Show that no fewer than t vectors in S can span S.

Solution Since S has dimension t, there is a basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_t\}$ for S having t vectors. If it were the case that there were fewer than t vectors in S that spanned S, say, $\text{Span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_s\} = S$, with $s < t$, then the theorem above would imply that $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_t$ are linearly dependent, which is not the case. Therefore, no fewer than t vectors in S can span S. □

7.3 Facts about dimension

In this section, we give some useful facts about dimension.

We have seen that the only subspaces of $\mathbf{R}^3$ are the following:

$$\{\mathbf{0}\}, \quad \text{lines through origin}, \quad \text{planes through origin}, \quad \mathbf{R}^3.$$

The dimensions of these subspaces are 0, 1, 2, and 3 (in turn). This illustrates the next theorem, which says that each subspace of $\mathbf{R}^n$ has dimension at most n.

> THEOREM. *If S is a subspace of $\mathbf{R}^n$, then S has a basis and $\dim S \leq n$.*

Proof. Let S be a subspace of $\mathbf{R}^n$. Since $\dim \mathbf{R}^n = n$, there is a set of n vectors that spans $\mathbf{R}^n$ (namely, a basis). Therefore, any linearly independent collection of vectors in S must consist of at most n vectors by the first theorem of Section 7.2. Let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ be linearly independent vectors in S with s as large as possible and note that $s \leq n$ by the previous observation. We claim that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is a basis for S.

(Span$\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} = S$?) First, since S is a subspace, any linear combination of vectors in S is still in S, so Span$\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} \subseteq S$. Next, we prove the other inclusion. Let $\mathbf{x}$ be any vector in S. The vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s, \mathbf{x}$ cannot be linearly independent since there are $s + 1$ vectors here and we assumed that the largest number of linearly independent vectors in S is s. Therefore, there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_{s+1}$, not all zero, such that

$$\alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_s \mathbf{b}_s + \alpha_{s+1} \mathbf{x} = \mathbf{0}.$$

Now α_{s+1} cannot be zero, since, if it were, then the last term would go away and one of the scalars $\alpha_1, \alpha_2, \dots, \alpha_s$ would have to be nonzero, which would violate the assumption that the $\mathbf{b}_i$'s are linearly independent. Therefore, we can solve the above equation for $\mathbf{x}$ (the last step being the division of both sides by the nonzero number α_{s+1}). This expresses $\mathbf{x}$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$. We conclude that Span$\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} = S$.

($\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ linearly independent?) The vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ are linearly independent by the way we chose them.

Therefore, $\mathcal{B}$ is a basis for S, so $\dim S = s \leq n$ and the proof is complete. □

The following theorem says that a spanning set of a subspace can be reduced to a basis, and that a collection of linearly independent vectors in a subspace can be expanded to a basis. The theorem also says how to carry out these procedures.

THEOREM. *Let S be a subspace of $\mathbf{R}^n$ and let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ be vectors in S.*

(i) *If* Span$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s\} = S$*, then a basis for S can be obtained as follows: Starting from the left, if any $\mathbf{x}_i$ is in the span of the vectors that come before it, then remove $\mathbf{x}_i$ from the list and continue to the end of the list.*

(ii) *If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ are linearly independent, then a basis for S can be obtained as follows: If some vector $\mathbf{x}$ in S is not in the span of the vectors in the list, then add it to the list and repeat until the number of vectors in the list is the same as* $\dim S$.

Proof. We give only the main ideas of the proof.

(i) An argument similar to that given in Example 5.3.2 shows that if $\mathbf{x}_i$ is in the span of the vectors that come before it, then it can be removed without changing the span. Exercise 7–3 shows why the final list of vectors is linearly independent and hence a basis for their span.

(ii) An argument similar to the solution to Exercise 5–5 shows that if a list of vectors is linearly independent and a vector not in the span of those vectors is added to the list, then the new list is linearly independent. Since S is spanned by t vectors, where $t = \dim S$, the process cannot produce a list of more than t linearly independent vectors (by the first theorem of Section 7.2). Therefore, when the list has t vectors, these vectors must span S and hence form a basis. □

By convention, the span of the empty set is $\{\mathbf{0}\}$. Therefore, in part (i), if the first vector is $\mathbf{0}$, then it is removed, and if it is not $\mathbf{0}$, then it is retained.

7.3.1 Example Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ that forms a basis for $S = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution We follow the procedure in part (i) of the theorem. First, $\mathbf{x}_1$ is not $\mathbf{0}$ so it is retained. Next, $\mathbf{x}_2 = -2\mathbf{x}_1$, so $\mathbf{x}_2$ is removed from the list. Next, $\mathbf{x}_3$ is not a linear combination (i.e., multiple in this case) of $\mathbf{x}_1$, so it is retained in the list. Finally, by inspection (or solving a system) we see that $\mathbf{x}_4 = (-1)\mathbf{x}_1 + 1\mathbf{x}_3$, so $\mathbf{x}_4$ is removed from the list. Therefore, $\{\mathbf{x}_1, \mathbf{x}_3\}$ is a basis for S. □

7.3.2 Example If possible, find a basis for $\mathbf{R}^4$ containing the vectors $\mathbf{x}_1 = [2, 5, 0, 0]^T$ and $\mathbf{x}_2 = [-1, 0, 3, 0]^T$.

Solution First, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, since neither is a linear combination (i.e., multiple in this case) of the other, so part (ii) of the theorem guarantees that such a basis can be found and we follow the procedure.

We claim that the vector $\mathbf{x}_3 = [1, 0, 0, 0]^T$ is not in $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$. Suppose, to the contrary, that we had

$$\mathbf{x}_3 = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2,$$

that is,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Looking at the third components, we see that α_2 must be zero so the last term goes away. But then, looking at the second components, we see that α_1 must be zero. But then the equation says that $\mathbf{x}_3 = \mathbf{0}$, which is not the case. Therefore, $\mathbf{x}_3$ is not in $\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ and we add it to the list to get $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

Next, the vector $\mathbf{x}_4 = [0,0,0,1]^T$ is not in $\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, since the fourth component in each of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ is zero, so there is no way to form a linear combination of these vectors to get the required fourth component 1 in $\mathbf{x}_4$. Therefore, we add $\mathbf{x}_4$ to the list.

The list is now $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. Since $\dim \mathbf{R}^4 = 4$, the process stops and we conclude that

$$\{[2,5,0,0]^T, [-1,0,3,0]^T, [1,0,0,0]^T, [0,0,0,1]^T\}$$

is a basis for $\mathbf{R}^4$.

(This is not the only possible solution.) □

THEOREM. *Let S be a subspace of $\mathbf{R}^n$ of dimension s. If $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ are s vectors in S, then $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is a basis for S if* either *of the following holds:*

(i) $\operatorname{Span}\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} = S$,

(ii) $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ *are linearly independent.*

Proof. Assume that (i) holds, so that $\operatorname{Span}\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} = S$. By the preceding theorem, one can obtain a basis for S by removing vectors, if necessary, from the list $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$. However, since any basis for S must have s vectors, no vectors can be removed, that is, $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is already a basis for S.

Showing that $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is a basis if (ii) holds is left as an exercise (see Exercise 7–6). □

The theorem says that if you know in advance that a subspace has dimension s, then you can tell whether a set of s vectors in S is a basis by checking only one of the basis properties.

7.3.3 Example Show that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$, where

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 9 \end{bmatrix}.$$

Solution The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent since neither is a linear combination (i.e., multiple in this case) of the other. Since $\mathbf{R}^2$ has dimension two, the theorem (with $S = \mathbf{R}^2$) says that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$. □

7.4 Subspaces associated with a matrix

NULL SPACE, ROW SPACE, COLUMN SPACE.

Let $\mathbf{A}$ be an $m \times n$ matrix.

- The **null space of A** (denoted $\operatorname{Null} \mathbf{A}$) is the set of all vectors $\mathbf{x}$ in $\mathbf{R}^n$ for which $\mathbf{A}\mathbf{x} = \mathbf{0}$.
- The **row space of A** (denoted $\operatorname{Row} \mathbf{A}$) is the subspace of $\mathbf{R}^n$ spanned by the rows of $\mathbf{A}$ (written as columns).
- The **column space of A** (denoted $\operatorname{Col} \mathbf{A}$) is the subspace of $\mathbf{R}^m$ spanned by the columns of $\mathbf{A}$.

THEOREM. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are matrices with* $\mathbf{A} \sim \mathbf{B}$*, then* $\operatorname{Row} \mathbf{A} = \operatorname{Row} \mathbf{B}$.

Proof. Assume first that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying a single row operation. If that row operation is of type I (interchange two rows), then $\operatorname{Row} \mathbf{A} = \operatorname{Row} \mathbf{B}$. If that row operation is of type II or III, then the replaced row is a linear combination of the rows of $\mathbf{A}$ so that $\operatorname{Row} \mathbf{B} \subseteq \operatorname{Row} \mathbf{A}$. By repeating this for each applied row operation we see that if $\mathbf{A} \sim \mathbf{B}$, then $\operatorname{Row} \mathbf{B} \subseteq \operatorname{Row} \mathbf{A}$. But $\mathbf{A} \sim \mathbf{B}$ implies that $\mathbf{B} \sim \mathbf{A}$ so that $\operatorname{Row} \mathbf{A} \subseteq \operatorname{Row} \mathbf{B}$ as well. □

THEOREM. *If* $\mathbf{A}$ *is a matrix and* $\mathbf{B}$ *is a row echelon form of* $\mathbf{A}$*, then the nonzero rows of* $\mathbf{B}$ *(written as columns) form a basis for* $\operatorname{Row} \mathbf{A}$.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be as stated. The nonzero rows of $\mathbf{B}$ are linearly independent since the matrix having these rows as columns has full rank. Therefore, these rows form a basis for their span $\operatorname{Row} \mathbf{B}$, which is the same as $\operatorname{Row} \mathbf{A}$ by the previous theorem. □

THEOREM. *For any matrix* $\mathbf{A}$ *we have,* $\dim \operatorname{Col} \mathbf{A} = \dim \operatorname{Row} \mathbf{A}$.

Proof. Let $\mathbf{B}$ be a row echelon form of $\mathbf{A}$. The columns of $\mathbf{A}$ corresponding to the pivot columns of $\mathbf{B}$ are linearly independent by the theorem before Example

5.2.3. Since the number p of these pivot columns is the same as the number of nonzero rows of $\mathbf{B}$, which is the dimension of $\operatorname{Row}\mathbf{A}$ by the preceding theorem, we have $\dim\operatorname{Col}\mathbf{A} \geq p = \dim\operatorname{Row}\mathbf{A}$. Applying this inequality to $\mathbf{A}^T$, we get $\dim\operatorname{Row}\mathbf{A} = \dim\operatorname{Col}\mathbf{A}^T \geq \dim\operatorname{Row}\mathbf{A}^T = \dim\operatorname{Col}\mathbf{A}$. Therefore, $\dim\operatorname{Col}\mathbf{A} = \dim\operatorname{Row}\mathbf{A}$. □

7.4.1 Example Find a basis for each of the spaces $\operatorname{Null}\mathbf{A}$, $\operatorname{Row}\mathbf{A}$, and $\operatorname{Col}\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 1 & 6 \\ -1 & -2 & 1 & 5 \end{bmatrix}.$$

Solution ($\operatorname{Null}\mathbf{A}$) The equation $\mathbf{Ax} = \mathbf{0}$ corresponds to a system with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2 & -4 & 1 & 6 & 0 \\ -1 & -2 & 1 & 5 & 0 \end{array}\right] \begin{matrix} 2 \; 1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right] \begin{matrix} \\ -1 \\ \\ \end{matrix}$$
$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so $\operatorname{Null}\mathbf{A} = \{[-2t+s, t, -4s, s]^T \mid t, s \in \mathbf{R}\}$. Since

$$\begin{bmatrix} -2t+s \\ t \\ -4s \\ s \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$

the vectors $[-2,1,0,0]^T$ and $[1,0,-4,1]^T$ span $\operatorname{Null}\mathbf{A}$. Also, they are linearly independent. (In fact, this method of writing the general solution as a linear combination will always yield linearly independent vectors.) Therefore $\{[-2,1,0,0]^T, [1,0,-4,1]^T\}$ is a basis for $\operatorname{Null}\mathbf{A}$.

($\operatorname{Row}\mathbf{A}$) By the second theorem above, the nonzero rows in a row echelon form of $\mathbf{A}$ form a basis for $\operatorname{Row}\mathbf{A}$, so $\{[1,2,0,-1]^T, [0,0,1,4]^T\}$ is a basis for $\operatorname{Row}\mathbf{A}$.

($\operatorname{Col}\mathbf{A}$) By the third theorem above, $\dim\operatorname{Col}\mathbf{A}$ is the same as $\dim\operatorname{Row}\mathbf{A}$, which is two by what we just showed. Therefore, any two linearly independent vectors in $\operatorname{Col}\mathbf{A}$ will form a basis.

The first and third columns of the row echelon form of $\mathbf{A}$ that we found above are the pivot columns. These columns might not be in $\operatorname{Col}\mathbf{A}$, since row equivalent matrices do not necessarily have the same column space. However, the corresponding columns of the original matrix $\mathbf{A}$, namely $[1,-2,-1]^T$ and $[0,1,1]^T$ *are* in $\operatorname{Col}\mathbf{A}$. Moreover, these columns are linearly independent. (In fact, the columns of the original matrix corresponding to the pivot columns in a row echelon form of the matrix will always be linearly independent since the matrix they form will have full rank.) Therefore, $\{[1,-2,-1]^T, [0,1,1]^T\}$ is a basis for Col **A**. □

We summarize the steps in the preceding solution:

FINDING BASES FOR NULL A, ROW A, AND COL A.

- (Null **A**) Solve $\mathbf{Ax} = \mathbf{0}$, write general solution as linear combination by grouping like terms, and use resulting vectors as basis.
- (Row **A**) Find row-reduced form of **A** and use its *nonzero* rows (written as columns) as basis.
- (Col **A**) Find row-reduced form of **A** and use for basis the columns *of the original matrix* **A** corresponding to the pivot columns in the row-reduced matrix.

7.4.2 Example Find a basis for each of the spaces Null **A**, Row **A**, and Col **A**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 4 \\ -2 & -2 & 3 & 0 & -9 \end{bmatrix}.$$

Solution (Null **A**) Solving $\mathbf{Ax} = \mathbf{0}$, we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 \\ -2 & -2 & 3 & 0 & -9 & 0 \end{array}\right] \begin{matrix} -1 & 2 \end{matrix} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & -3 & 0 \end{array}\right] \begin{matrix} -1 & -3 \end{matrix}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so Null $\mathbf{A} = \{[-t-3s, t, s, -2s, s]^T \mid t, s \in \mathbf{R}\}$. Since

$$\begin{bmatrix} -t-3s \\ t \\ s \\ -2s \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix},$$

the set $\{[-1, 1, 0, 0, 0]^T, [-3, 0, 1, -2, 1]^T\}$ is a basis for Null $\mathbf{A}$.

(Row $\mathbf{A}$) $\{[1, 1, 0, 0, 3]^T, [0, 0, 1, 0, -1]^T, [0, 0, 0, 1, 2]^T\}$ is a basis for Row $\mathbf{A}$.

(Col $\mathbf{A}$) Columns one, three, and four are the pivot columns, so

$$\{[1, 0, 1, -2]^T, [0, 1, 1, 3]^T, [0, 0, 1, 0]^T\}$$

is a basis for Col $\mathbf{A}$. □

The dimension of the row space of a matrix is called the **rank** of the matrix. By the third theorem of this section, this number is also the dimension of the column space of the matrix:

$$\operatorname{rank} \mathbf{A} = \dim \operatorname{Row} \mathbf{A} \ (= \dim \operatorname{Col} \mathbf{A}).$$

The dimension of the null space of a matrix is called the **nullity** of the matrix:

$$\operatorname{nullity} \mathbf{A} = \dim \operatorname{Null} \mathbf{A}.$$

RANK + NULLITY THEOREM.

If $\mathbf{A}$ is an $m \times n$ matrix, then

$$\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n.$$

Proof. Let $\mathbf{A}$ be an $m \times n$ matrix. The rank of $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$, which is the number of pivot columns in a row-echelon form of $\mathbf{A}$ (see Example 7.4.1). The nullity of $\mathbf{A}$ is the dimension of the null space of $\mathbf{A}$, which is the number of nonpivot columns in a row-echelon form of $\mathbf{A}$ (again, see Example 7.4.1). Therefore, $\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A}$ is the total number of columns of $\mathbf{A}$, which is n. □

7.4.3 Example Verify the Rank + Nullity theorem using the matrix $\mathbf{A}$ of Example 7.4.2.

Solution Referring to the solution to Example 7.4.2, we have

$$\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = 3 + 2 = 5 = n,$$

so the Rank + Nullity theorem is verified. □

> THEOREM. *Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear function and let $\mathbf{A}$ be the matrix of L (so that $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$). We have*
>
> (i) $\operatorname{im} L = \operatorname{Col} \mathbf{A}$,
>
> (ii) $\ker L = \operatorname{Null} \mathbf{A}$.
>
> (iii) $\dim \operatorname{im} L + \dim \ker L = n$

Proof. (i) The product $\mathbf{A}\mathbf{x}$ can be interpreted as the linear combination of the columns of $\mathbf{A}$ with scalar factors given by the entries in $\mathbf{x}$. Therefore,

$$\operatorname{im} L = \{L(\mathbf{x}) \,|\, \mathbf{x} \in \mathbf{R}^n\} = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \mathbf{R}^n\} = \operatorname{Col} \mathbf{A}.$$

(ii) This follows directly from the definitions:

$$\ker L = \{\mathbf{x} \in \mathbf{R}^n \,|\, L(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbf{R}^n \,|\, \mathbf{A}\mathbf{x} = \mathbf{0}\} = \operatorname{Null} \mathbf{A}.$$

(iii) Using parts (i) and (ii) and the Rank + Nullity theorem, we have

$$\dim \operatorname{im} L + \dim \ker L = \dim \operatorname{Col} \mathbf{A} + \dim \operatorname{Null} \mathbf{A} = \operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n.$$

□

The equation in part (iii) can be written

$$\dim \operatorname{im} L = \dim \mathbf{R}^n - \dim \ker L$$

and interpreted as saying that the number of degrees of freedom in the space of outputs of L is that in the space of inputs less that lost by being sent to $\mathbf{0}$.

7.4.4 Example Let $L : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear function given by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + x_3 \\ 5x_1 + 5x_2 + 5x_3 \end{bmatrix}.$$

(a) Find a basis for $\ker L$.

(b) Find a basis for $\operatorname{im} L$.

(c) Verify part (iii) of the previous theorem.

Solution

(a) By the previous theorem, $\ker L = \operatorname{Null} \mathbf{A}$ where $\mathbf{A}$ is the matrix of L. We have

$$\mathbf{A} = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & L(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \end{bmatrix}.$$

Solving $\mathbf{Ax} = \mathbf{0}$ we are led to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right] -5 \curvearrowright \quad \sim \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

so $\ker L = \operatorname{Null} \mathbf{A} = \{[-t-s, t, s]^T \mid t, s \in \mathbf{R}\}$. Since

$$\begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$\{[-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a basis for $\ker L$.

(b) By the previous theorem, $\operatorname{im} L = \operatorname{Col} \mathbf{A}$, so $\{[1, 5]^T\}$ is a basis for $\operatorname{im} L$.

(c) We have $\dim \operatorname{im} L + \dim \ker L = 1 + 2 = 3 = n$, so part (iii) of the theorem is verified.

□

7 – Exercises

7–1 Find the dimension of $S = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

7–2 Let S and T be subspaces of $\mathbf{R}^n$ with $T \subseteq S$. Show that $\dim T \leq \dim S$.

HINT: By the first theorem of Section 7.3, S has a basis $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ and T has a basis $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_t\}$. Now use the first theorem of Section 7.2.

7–3 Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ be vectors in $\mathbf{R}^n$ and assume that none is in the span of the vectors that come before it in the list. Show that the vectors are linearly independent.

HINT: You are being asked to provide a portion of the proof of the theorem before Example 7.3.1. Argue directly from the definition of linear independence. Suppose that $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4\mathbf{x}_4 = \mathbf{0}$. Show that if α_4 is not zero, then a contradiction arises. Therefore, $\alpha_4 = 0$ and the last term on the left goes away. Repeat the argument to show that $\alpha_i = 0$ for all i.

7–4 Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ that forms a basis for $S = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} -2 \\ 9 \\ -2 \end{bmatrix}.$$

7–5 If possible, find a basis for $\mathbf{R}^4$ containing the vectors $\mathbf{x}_1 = [1, 0, 0, 1]^T$ and $\mathbf{x}_2 = [2, 1, -3, 0]^T$.

7–6 Let S be a subspace of $\mathbf{R}^n$ of dimension s and let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ be s linearly independent vectors in S. Show that $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is a basis for S.

HINT: You are being asked to complete the proof of the theorem before Example 7.3.3. Use an argument similar to that given in the first part of the proof.

7–7 Show that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $\mathbf{R}^3$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -3 \\ -6 \\ 4 \end{bmatrix}.$$

HINT: Use the theorem before Example 7.3.3.

7–8 Find a basis for each of the spaces Null $\mathbf{A}$, Row $\mathbf{A}$, and Col $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 2 & 4 \\ -1 & -2 & -5 & -2 & -3 \\ 0 & 1 & 3 & -2 & 8 \\ 2 & 3 & 7 & 6 & 0 \end{bmatrix}.$$

7–9 Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear function given by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ -3x_1 - 6x_2 + x_3 \\ 2x_1 + 4x_2 \end{bmatrix}.$$

(a) Find a basis for $\ker L$.

(b) Find a basis for $\operatorname{im} L$.

(c) Verify part (iii) of the theorem before Example 7.4.4 .