14 Diagonalization

14.1 Change of basis

Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be an ordered basis for $\mathbf{R}^2$ and let $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}$ (the matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$ as columns). If $\mathbf{x}$ is a vector in $\mathbf{R}^2$, then its coordinate vector $[\mathbf{x}]_\mathcal{B}$ relative to $\mathcal{B}$ satisfies the formula

$$\boxed{\mathbf{B}[\mathbf{x}]_\mathcal{B} = \mathbf{x}}$$

We can see this by writing $[\mathbf{x}]_\mathcal{B} = [\alpha_1, \alpha_2]^T$ to get

$$\mathbf{B}[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 = \mathbf{x}.$$

The columns of $\mathbf{B}$ are linearly independent (since they form a basis) so $\mathbf{B}$ has full rank and is therefore invertible. This allows us to solve the equation above for the coordinate vector:

$$\boxed{[\mathbf{x}]_\mathcal{B} = \mathbf{B}^{-1}\mathbf{x}}$$

14.1.1 Example Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be the ordered basis of $\mathbf{R}^2$ with $\mathbf{b}_1 = [1, 2]^T$ and $\mathbf{b}_2 = [2, 3]^T$. Use the formula above to find the coordinate vector $[\mathbf{x}]_\mathcal{B}$ where $\mathbf{x} = [1, -1]^T$.

Solution We have

$$\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix}1 & 2 \\ 2 & 3\end{bmatrix},$$

so

$$[\mathbf{x}]_\mathcal{B} = \mathbf{B}^{-1}\mathbf{x} = \frac{1}{-1}\begin{bmatrix}3 & -2 \\ -2 & 1\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} = -\begin{bmatrix}5 \\ -3\end{bmatrix} = \begin{bmatrix}-5 \\ 3\end{bmatrix}.$$

(Check: $(-5)\mathbf{b}_1 + (3)\mathbf{b}_2 = -5\begin{bmatrix}1 \\ 2\end{bmatrix} + 3\begin{bmatrix}2 \\ 3\end{bmatrix} = \begin{bmatrix}1 \\ -1\end{bmatrix} = \mathbf{x}$.) □

Let $\mathcal{C} = (\mathbf{c}_1, \mathbf{c}_2)$ be another ordered basis for $\mathbf{R}^2$ and put $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix}$. For any vector $\mathbf{x}$ in $\mathbf{R}^2$, the coordinate vectors of $\mathbf{x}$ relative to $\mathcal{B}$ and $\mathcal{C}$ satisfy the equation

$$\boxed{\mathbf{B}[\mathbf{x}]_\mathcal{B} = \mathbf{C}[\mathbf{x}]_\mathcal{C}}$$

This equation holds since both sides equal $\mathbf{x}$ by the first equation of the section. We can solve for $[\mathbf{x}]_{\mathcal{B}}$ to get

$$\boxed{[\mathbf{x}]_{\mathcal{B}} = \underbrace{\mathbf{B}^{-1}\mathbf{C}}_{\mathbf{P}}[\mathbf{x}]_{\mathcal{C}}}$$

The matrix $\mathbf{P} = \mathbf{B}^{-1}\mathbf{C}$ is called the "transformation matrix from $\mathcal{C}$ to $\mathcal{B}$."

14.1.2 Example Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ and $\mathcal{C} = (\mathbf{c}_1, \mathbf{c}_2)$ be the ordered bases of $\mathbf{R}^2$ with

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^2$.

(a) Find the transformation matrix $\mathbf{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

(b) Given that $[\mathbf{x}]_{\mathcal{C}} = [-3, 6]^T$, find $[\mathbf{x}]_{\mathcal{B}}$.

(c) Given that $[\mathbf{y}]_{\mathcal{B}} = [7, 2]^T$, find $[\mathbf{y}]_{\mathcal{C}}$.

Solution

(a) We have

$$\mathbf{P} = \mathbf{B}^{-1}\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{bmatrix}.$$

(b) We have

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ 15 \end{bmatrix}.$$

(c) Multiplying both sides of the change of basis formula by $\mathbf{P}^{-1}$ we get

$$[\mathbf{y}]_{\mathcal{C}} = \mathbf{P}^{-1}[\mathbf{y}]_{\mathcal{B}} = \frac{1}{\frac{1}{2}} \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 15 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 \\ 16 \end{bmatrix}.$$

□

Here is the general formulation for a change of basis:

CHANGE OF BASIS.

Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ and $\mathcal{C} = (\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n)$ be two ordered bases for $\mathbf{R}^n$, and put $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots \mathbf{b}_n\end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n\end{bmatrix}$. For any vector $\mathbf{x}$ in $\mathbf{R}^n$, we have

$$\mathbf{B}[\mathbf{x}]_{\mathcal{B}} = \mathbf{C}[\mathbf{x}]_{\mathcal{C}}$$

so that

$$[\mathbf{x}]_{\mathcal{B}} = \underbrace{\mathbf{B}^{-1}\mathbf{C}}_{\mathbf{P}}[\mathbf{x}]_{\mathcal{C}}.$$

The matrix $\mathbf{P} = \mathbf{B}^{-1}\mathbf{C}$ is called the **transformation matrix from $\mathcal{C}$ to $\mathcal{B}$**.

14.2 Linear function and basis change

THEOREM.

Let $L : \mathbf{R}^n \to \mathbf{R}^n$ be a linear function, let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for $\mathbf{R}^n$, let $\mathbf{P}$ be the transformation matrix from $\mathcal{C}$ to $\mathcal{B}$, and let $\mathbf{A}$ be the matrix of L relative to $\mathcal{B}$. The matrix of L relative to $\mathcal{C}$ is $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, that is,

$$[L(\mathbf{x})]_{\mathcal{C}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}[\mathbf{x}]_{\mathcal{C}}$$

for all $\mathbf{x} \in \mathbf{R}^n$.

Proof. For all $\mathbf{x} \in \mathbf{R}^n$ we have

$$\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P}[\mathbf{x}]_{\mathcal{C}} &= \mathbf{P}^{-1}\mathbf{A}[\mathbf{x}]_{\mathcal{B}} && \text{definition of transformation matrix} \\
&= \mathbf{P}^{-1}[L(\mathbf{x})]_{\mathcal{B}} && \mathbf{A} \text{ is matrix of } L \text{ relative to } \mathcal{B} \\
&= [L(\mathbf{x})]_{\mathcal{C}} && \mathbf{P}^{-1} \text{ is transformation matrix from } \mathcal{B} \text{ to } \mathcal{C}.
\end{aligned}$$

□

14.2.1 Example Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear function given by

$$L(\mathbf{x}) = \begin{bmatrix} \frac{1}{2}x_1 + \frac{3}{2}x_2 \\ \frac{3}{2}x_1 + \frac{1}{2}x_2 \end{bmatrix}.$$

(a) Find the matrix $\mathbf{A}$ of L (relative to the standard ordered basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$).

(b) Find the transformation matrix $\mathbf{P}$ from $\mathcal{C} = ([1,1]^T, [-1,1]^T)$ to $\mathcal{E}$.

(c) Use the theorem to find the matrix of L relative to $\mathcal{C}$.

Solution

(a) We have
$$\mathbf{A} = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) Writing $\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$ we see that $\mathbf{E} = \mathbf{I}$, so that
$$\mathbf{P} = \mathbf{E}^{-1}\mathbf{C} = \mathbf{I}^{-1}\mathbf{C} = \mathbf{C} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(c) According to the theorem, the matrix of L relative to $\mathcal{C}$ is
$$\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

□

In this example, the matrix of L relative to the basis $\mathcal{C}$ turned out to be much simpler than the matrix relative to the standard basis. The reason is that $\mathcal{C}$ is a basis for $\mathbf{R}^2$ consisting of eigenvectors of L. We investigate such bases in the next section.

14.3 Method for diagonalization

The matrix
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
is an example of a "diagonal matrix." In general, a **diagonal matrix** is a matrix having the property that every entry not on the main diagonal is 0.

An $n \times n$ matrix $\mathbf{A}$ is **diagonalizable** if there exists an invertible $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix.

THEOREM.

Let $\mathbf{A}$ be an $n \times n$ matrix. The matrix $\mathbf{A}$ is diagonalizable if and only if there exists a basis for $\mathbf{R}^n$ consisting of eigenvectors of $\mathbf{A}$. In this case, if $\mathbf{P}$ is the matrix with the eigenvectors as columns, then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

with $\mathbf{D}$ diagonal.

Proof. We will prove only the case $n = 2$. Assume that there exists a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbf{R}^2$ consisting of eigenvectors of $\mathbf{A}$. Let λ_1 and λ_2 be the corresponding eigenvalues and write $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. We have

$$\mathbf{AP} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{b}_1 & \lambda_2\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{PD}$$

Therefore, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Conversely, if $\mathbf{P}$ diagonalizes $\mathbf{A}$, then the equation above shows that the columns of $\mathbf{P}$ must be eigenvectors of $\mathbf{A}$ and these columns are linearly independent since $\mathbf{P}$ is invertible. □

14.3.1 Example Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. If possible, find an invertible 2×2 matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix.

Solution According to the theorem, such a matrix $\mathbf{P}$ exists if and only if there exists a basis for $\mathbf{R}^2$ consisting of eigenvectors of $\mathbf{A}$. The characteristic polynomial of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

The eigenvalues of $\mathbf{A}$ are the zeros of this polynomial, namely, $\lambda = 2, 3$.

Next, the λ-eigenspace of $\mathbf{A}$ is the solution set of the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$:

$(\lambda = 2)$

$$[\mathbf{A} - 2\mathbf{I} \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so the 2-eigenspace of $\mathbf{A}$ is $\{[t, t]^T \,|\, t \in \mathbf{R}\}$. Letting $t = 1$, we get a 2-eigenvector $[1, 1]^T$;

($\lambda = 3$)

$$[\mathbf{A} - 3\mathbf{I}\,|\,\mathbf{0}] \quad = \quad \left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

so the 3-eigenspace of $\mathbf{A}$ is $\{[\frac{t}{2}, t]^T \,|\, t \in \mathbf{R}\}$. Letting $t = 2$ (to avoid fractions), we get a 3-eigenvector $[1, 2]^T$.

The eigenvectors $[1, 1]^T$ and $[1, 2]^T$ of $\mathbf{A}$ form a basis for $\mathbf{R}^2$ (neither is a multiple of the other so they are linearly independent; since $\dim \mathbf{R}^2 = 2$, they form a basis). According to the theorem, the matrix $\mathbf{P}$ with these vectors as columns should have the indicated property:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Computing we get

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \mathbf{D}. \end{aligned}$$

(Note that the eigenvalues of $\mathbf{A}$ appear along the main diagonal of $\mathbf{D}$.) □

14.3.2 Example Is the matrix $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ diagonalizable? Explain.

Solution The characteristic polynomial of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

Since this polynomial has no zeros (in $\mathbf{R}$), the matrix $\mathbf{A}$ has no eigenvalues. Therefore, there is no basis for $\mathbf{R}^2$ consisting of eigenvectors of $\mathbf{A}$ and $\mathbf{A}$ is not diagonalizable according to the theorem.

(Here's another way to see that $\mathbf{A}$ has no eigenvalues. $\mathbf{A}$ is the matrix of the linear function "90° clockwise rotation." Since this function sends no nonzero vector to a multiple of itself, it has no eigenvalues, and therefore $\mathbf{A}$ has no eigenvalues either.) □

14.4 Power of matrix

It is often necessary to compute high powers of a square matrix $\mathbf{A}$, such as $\mathbf{A}^{10}$. Just multiplying the matrix $\mathbf{A}$ by itself over and over again can be quite tedious.

However, if $\mathbf{A}$ is diagonalizable, then there is an observation that greatly reduces the number of computations:

Let $\mathbf{A}$ be a diagonalizable matrix, so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ with $\mathbf{D}$ diagonal. Solving this equation for $\mathbf{A}$, we get $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Note that

$$\begin{aligned}\mathbf{A}^2 = \mathbf{A}\mathbf{A} &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = \mathbf{P}\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\mathbf{P}^{-1}\\ &= \mathbf{P}\mathbf{D}\mathbf{I}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1},\end{aligned}$$

and in less detail

$$\mathbf{A}^3 = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}.$$

In general,

$$\boxed{\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} \quad\Rightarrow\quad \mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}}$$

Note that since $\mathbf{D}$ is diagonal, the power $\mathbf{D}^n$ is obtained by raising each diagonal entry to the nth power.

14.4.1 Example Compute $\mathbf{A}^{10}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Solution In Example 14.3.1 we found a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ with $\mathbf{D}$ diagonal. Using the results of that example we get

$$\begin{aligned}\mathbf{A}^{10} = \mathbf{P}\mathbf{D}^{10}\mathbf{P}^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{10}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 2^{10} & 3^{10} \\ 2^{10} & 2\cdot 3^{10} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 2^{11} - 3^{10} & -2^{10} + 3^{10} \\ 2^{11} - 2\cdot 3^{10} & -2^{10} + 2\cdot 3^{10} \end{bmatrix}\\ &= \begin{bmatrix} -57001 & 58025 \\ -116050 & 117074 \end{bmatrix}\end{aligned}$$

□

14 – Exercises

14–1 Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ and $\mathcal{C} = (\mathbf{c}_1, \mathbf{c}_2)$ be the ordered bases of $\mathbf{R}^2$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

and let $\mathbf{x}$ be a vector in $\mathbf{R}^2$.

(a) Find the transformation matrix $\mathbf{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

(b) Given that $[\mathbf{x}]_\mathcal{C} = [1, 2]^T$, use part (a) to find $[\mathbf{x}]_\mathcal{B}$ and check your answer by showing that both coordinate vectors yield the same vector $\mathbf{x}$.

14–2 Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be "reflection across the line $x_2 = x_1/\sqrt{3}$" (the line through the origin making an angle of 30° with the x_1-axis).

(a) Find the matrix $\mathbf{A}$ of L (relative to the standard ordered basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2)$).

(b) Find the transformation matrix $\mathbf{P}$ from $\mathcal{C} = ([\sqrt{3}, 1]^T, [-1, \sqrt{3}]^T)$ to $\mathcal{E}$.

(c) Use the theorem in Section 14.2 to find the matrix of L relative to $\mathcal{C}$.

HINT: Recall the 30-60-90° triangle with legs 1, 2, and $\sqrt{3}$.

14–3 Let $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$.

(a) If possible, find an invertible 2×2 matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix.

(b) Find $\mathbf{A}^{10}$. (Hint: Use the formula in Section 14.4.)