12 Determinant

12.1 Definition

Let $\mathbf{A}$ be an $n \times n$ matrix. The "determinant" of $\mathbf{A}$, written $\det \mathbf{A}$, is a certain number associated to $\mathbf{A}$.

This number has some useful properties. For instance, the matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A}$ is nonzero.

Also, $|\det \mathbf{A}|$ is the volume of the parallelepiped formed by the columns of $\mathbf{A}$ (the "Jacobian" is a determinant that appears in an integral after a change of variables due to this property).

The determinant is defined recursively:

1×1 determinant

The determinant of a 1×1 matrix is the single entry itself:

$$\det \begin{bmatrix} a \end{bmatrix} = a.$$

2×2 determinant

The determinant of a 2×2 matrix is given by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For instance,

$$\det \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = (3)(2) - (-1)(4) = 10.$$

The determinant is also notated using vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

3×3 determinant

In order to define the determinant of a 3×3 matrix we need some terminology. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}.$$

The (i,j)**-minor** of $\mathbf{A}$, denoted m_{ij}, is the determinant of the matrix obtained from $\mathbf{A}$ by removing the ith row and the jth column. For instance,

$$m_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix}, \quad m_{12} = \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}, \quad m_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix},$$
$$m_{21} = \begin{vmatrix} -1 & 0 \\ 1 & 6 \end{vmatrix}, \quad \text{etc.}$$

These are the numbers 16, −4, −14, −6, etc.

The (i,j)**-cofactor** of $\mathbf{A}$, denoted c_{ij}, is the corresponding minor m_{ij} multiplied by the number $(-1)^{i+j}$:

$$c_{ij} = (-1)^{i+j} m_{ij}.$$

For instance,

$$\begin{aligned}
c_{11} &= (-1)^{1+1} m_{11} = (+1) \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = (+1)(16) = 16, \\
c_{12} &= (-1)^{1+2} m_{12} = (-1) \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = (-1)(-4) = 4, \\
c_{13} &= (-1)^{1+3} m_{13} = (+1) \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} = (+1)(-14) = -14, \\
c_{21} &= (-1)^{2+1} m_{21} = (-1) \begin{vmatrix} -1 & 0 \\ 1 & 6 \end{vmatrix} = (-1)(-6) = 6,
\end{aligned}$$

etc.

Instead of computing $(-1)^{i+j}$, it is often easier just to use the fact that it is either +1 or −1 with the sign given by a checkerboard pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

The determinant of the 3×3 matrix $\mathbf{A}$ is (using the above computations)

$$\begin{aligned}
\det \mathbf{A} &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\
&= (2)(16) + (-1)(4) + (0)(-14) \\
&= 28.
\end{aligned}$$

(a_{ij} is the entry in the ith row and jth column of $\mathbf{A}$).

In words, the determinant is computed by multiplying each entry in the first row by its corresponding cofactor and adding the results. This is called computing the determinant by "expanding along the first row."

It is a fact that the determinant can be computed by expanding along any row or any column (with the same results). For instance, expanding along the second column we get

$$\det \mathbf{A} = a_{12}c_{12} + a_{22}c_{22} + a_{23}c_{23},$$

so

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix} = -1(-1)\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + 3(+1)\begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} + 1(-1)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= (-4) + 3(12) - (4)$$
$$= 28$$

(same as before).

It is usually the best strategy to expand along a row or column with the greatest number of zeros.

12.1.1 Example Find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 5 & 0 \\ -7 & 8 & 3 \end{bmatrix}.$$

Solution We expand along the third column (due to the zeros) and get

$$\det \mathbf{A} = \begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & 0 \\ -7 & 8 & 3 \end{vmatrix}$$
$$= 0 + 0 + 3(+1)\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}$$
$$= 3(6) = 18.$$

□

n × n determinant

The determinant of an $n \times n$ matrix is defined just like the determinant of a 3×3 matrix: choose any row or column, multiply its entries by their corresponding cofactors, and add the results.

12.1.2 Example Find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

Solution We expand along the fourth row (due to the zeros) and for the resulting 3×3 determinants, we expand along the first and second rows, respectively:

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} \\
&= 0 + 2(+1)\begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} + 0 + 1(+1)\begin{vmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & -2 \end{vmatrix} \\
&= 2\left(2(+1)\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + 2(-1)\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + 1(+1)\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}\right) \\
&\quad + 1\left(3(-1)\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + 0 + 1(-1)\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}\right) \\
&= 2\Big(2(3) - 2(4) + (-5)\Big) + 1\Big(- 3(-6) - (5)\Big) \\
&= -1.
\end{aligned}$$

□

12.2 Inverse matrix and determinant

Let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ -3 & 1 & 3 \end{bmatrix}.$$

The **cofactor matrix** of **A** is

$$\begin{aligned}
\mathbf{C} &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \\
&= \begin{bmatrix} +\begin{vmatrix} 6 & 8 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & 8 \\ -3 & 3 \end{vmatrix} & +\begin{vmatrix} -3 & 6 \\ -3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} -1 & 3 \\ -3 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 6 & 8 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ -3 & 8 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ -3 & 6 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 10 & -15 & 15 \\ -3 & 6 & -5 \\ -2 & -1 & 0 \end{bmatrix}.
\end{aligned}$$

The **adjoint of A**, denoted Adj $\mathbf{A}$ is the transpose of this cofactor matrix:

$$\operatorname{Adj}\mathbf{A} = \mathbf{C}^T = \begin{bmatrix} 10 & -3 & -2 \\ -15 & 6 & -1 \\ 15 & -5 & 0 \end{bmatrix}.$$

If we multiply $\mathbf{A}$ and Adj $\mathbf{A}$, we get

$$\begin{aligned} \mathbf{A}(\operatorname{Adj}\mathbf{A}) &= \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 10 & -3 & -2 \\ -15 & 6 & -1 \\ 15 & -5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= 5\,\mathbf{I}. \end{aligned}$$

The number 5 turns out to be the determinant of $\mathbf{A}$ (as the reader can check). Therefore, we have

$$\boxed{\mathbf{A}(\operatorname{Adj}\mathbf{A}) = (\det\mathbf{A})\mathbf{I}}$$

This formula holds in general and shows that Adj $\mathbf{A}$ is nearly an inverse of $\mathbf{A}$; we just need to divide it by det $\mathbf{A}$, provided this determinant is nonzero.

Theorem. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\det\mathbf{A} \neq 0$, *then* $\mathbf{A}$ *is invertible and*

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\operatorname{Adj}\mathbf{A}.$$

Continuing to work with $\mathbf{A}$ as above, we get

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\operatorname{Adj}\mathbf{A} = \frac{1}{5}\begin{bmatrix} 10 & -3 & -2 \\ -15 & 6 & -1 \\ 15 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{5} & -\frac{2}{5} \\ -3 & \frac{6}{5} & -\frac{1}{5} \\ 3 & -1 & 0 \end{bmatrix}.$$

12.2.1 Example Use this theorem to find the inverse of the general 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

assuming $\det\mathbf{A} \neq 0$.

Solution The cofactor matrix of $\mathbf{A}$ is

$$\mathbf{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix},$$

so the adjoint matrix of $\mathbf{A}$ is

$$\operatorname{Adj}\mathbf{A} = \mathbf{C}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Therefore,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{Adj}\mathbf{A} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(in agreement with the formula given in Section 11.4). □

12.3 Properties

An $n \times n$ matrix is **triangular** if it has either of the following forms:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \qquad \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

(illustrated here using 4×4 matrices).

The **main diagonal entries** of an $n \times n$ matrix $\mathbf{A}$ are the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

> THEOREM. *The determinant of a triangular matrix is the product of its main diagonal entries.*

The following example illustrates why this is the case:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} &= 1(+1) \begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} + 0 + 0 + 0 \\ &= 1 \left(5(+1) \begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} + 0 + 0 \right) \\ &= (1)(5)(8)(10) = 400. \end{aligned}$$

The next theorem says that the determinant of a product is the product of the determinants.

> THEOREM. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *matrices, then*
>
> $$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$

12.3.1 Example Verify the theorem using the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}.$$

Solution We have

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \qquad \det \mathbf{B} = \begin{vmatrix} 5 & 1 \\ 0 & 3 \end{vmatrix} = 15$$

and

$$\det(\mathbf{AB}) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}\right) = \begin{vmatrix} 5 & 7 \\ 15 & 15 \end{vmatrix} = -30,$$

so

$$\det(\mathbf{AB}) = -30 = (-2)(15) = (\det \mathbf{A})(\det \mathbf{B})$$

and the theorem is verified. □

THEOREM. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, then* $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$.

Proof. Let $\mathbf{A}$ be an $n \times n$ matrix.

($\Rightarrow$) Assume that $\mathbf{A}$ is invertible so that $\mathbf{A}^{-1}$ exists. By the preceding theorem,

$$(\det \mathbf{A})(\det \mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det \mathbf{I} = 1$$

($\mathbf{I}$ is triangular). Therefore, $\det \mathbf{A} \neq 0$.

($\Leftarrow$) Assume that $\det \mathbf{A} \neq 0$. By the theorem of Section 12.2, $\mathbf{A}$ is invertible. □

THEOREM. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix, then* $\det \mathbf{A}^T = \det \mathbf{A}$.

Proof. Let $\mathbf{A}$ be an $n \times n$ matrix. Since the columns of $\mathbf{A}^T$ are the rows of $\mathbf{A}$, and since a determinant can be computed by expanding along either a row or a column, the claim follows. □

THEOREM. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix and either its rows or columns are linearly dependent, then* $\det \mathbf{A} = 0$.

Proof. Let $\mathbf{A}$ be an $n \times n$ matrix and assume that its columns are linearly dependent. Then $\mathbf{A}$ does not have full rank (5.2), so it is not invertible (11.5). Therefore, $\det \mathbf{A} = 0$ by the previous theorem.

If the rows of $\mathbf{A}$ are linearly dependent, then the columns of $\mathbf{A}^T$ are linearly dependent, so, by what we have just shown, $\det \mathbf{A} = \det \mathbf{A}^T = 0$. □

12.3.2 Example Use inspection to find the determinant of each of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{bmatrix}$,

(b) $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$,

(c) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution

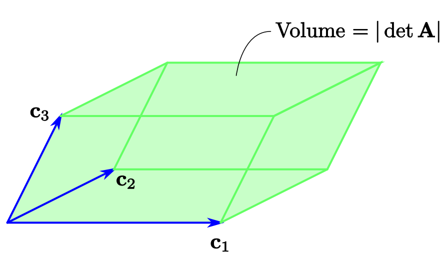
(a) $\det \mathbf{A} = 0$ (either expand along the third column or use the theorem).

(b) $\det \mathbf{B} = (1)(3)(6) = 18$ ($\mathbf{B}$ is triangular).

(c) $\det \mathbf{C} = 0$ (the third column is the sum of the other two so the columns are linearly dependent).

□

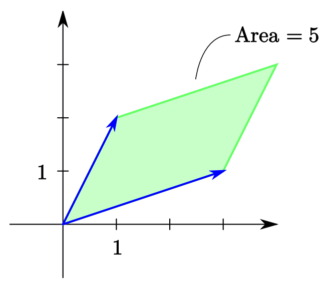
Theorem. *Let $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ be vectors in $\mathbf{R}^3$. The volume of the parallelepiped determined by these three vectors is $|\det \mathbf{A}|$, where $\mathbf{A}$ is the matrix having the three vectors as columns:*



A proof of this is given in calculus where the indicated determinant is called the scalar triple product of the vectors. It is this property of the determinant that gives rise to the "Jacobian" in an integral after a change of variables.

The two dimensional analog holds as well. It says that the area of the parallelogram determined by two vectors in $\mathbf{R}^2$ is $|\det \mathbf{A}|$, where $\mathbf{A}$ is the matrix having the two vectors as columns. For instance, the area of the parallelogram determined by the vectors $[3,1]^T$ and $[1,2]^T$ is

$$|\det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}| = 5$$



12.4 Effect of row operations

THEOREM. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. If the matrix* $\mathbf{B}$ *is obtained from* $\mathbf{A}$...

...*by (I) interchanging two rows, then* $\det \mathbf{B} = -\det \mathbf{A}$,
...*by (II) multiplying a row by* c*, then* $\det \mathbf{B} = c \det \mathbf{A}$,
...*by (III) adding a multiple of one row to another row, then* $\det \mathbf{B} = \det \mathbf{A}$.

Proof. In each case, we can write $\mathbf{B} = \mathbf{EA}$ where $\mathbf{E}$ is the corresponding elementary matrix, namely, the matrix obtained from the identity matrix $\mathbf{I}$ by applying the indicated row operation (see Exercise 2–3). By expanding repeatedly along unaffected rows (much like we did in the example illustrating the first theorem of 12.3) we eventually arrive at the following equalities in the three cases:

$$\text{(I)} \quad \det \mathbf{E} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1,$$

$$\text{(II)} \quad \det \mathbf{E} = \det \begin{bmatrix} c \end{bmatrix} = c,$$

$$\text{(III)} \quad \det \mathbf{E} = \det \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = 1.$$

Since $\det \mathbf{B} = \det \mathbf{EA} = (\det \mathbf{E})(\det \mathbf{A})$, the theorem follows. □

The second statement (II) can be interpreted as saying that a number c can be factored out of a row of the matrix. For instance,

$$\begin{vmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

This theorem gives us a way to compute a determinant that usually requires fewer steps than expanding along a row or column. The strategy is to apply row operations to put the matrix in triangular form (recording changes in the determinant along the way) and then just take the product of the main diagonal entries.

12.4.1 Example Find the determinant of the following matrix by first applying row operations to obtain a triangular matrix

$$\mathbf{A} = \begin{bmatrix} 14 & -7 & 28 \\ -2 & 1 & -1 \\ -6 & 8 & -10 \end{bmatrix}.$$

Solution We have

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 14 & -7 & 28 \\ -2 & 1 & -1 \\ -6 & 8 & -10 \end{vmatrix} && \text{(factor 7 out of first row)} \\
&= 7 \begin{vmatrix} 2 & -1 & 4 \\ -2 & 1 & -1 \\ -6 & 8 & -10 \end{vmatrix} \begin{matrix} 1 \curvearrowright & 3 \curvearrowright \end{matrix} && \text{(no change)} \\
&= 7 \begin{vmatrix} 2 & -1 & 4 \\ 0 & 0 & 3 \\ 0 & 5 & 2 \end{vmatrix} \updownarrow && \text{(sign change)} \\
&= -7 \begin{vmatrix} 2 & -1 & 4 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{vmatrix} \\
&= -7(2)(5)(3) = -210.
\end{aligned}$$

□

Caution: Since a type IV row operation is a combination of types II and III, it changes the determinant. For instance,

$$\begin{vmatrix} -3 & 1 \\ 2 & 5 \end{vmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \curvearrowright \quad = \quad \frac{1}{3} \begin{vmatrix} -3 & 1 \\ 0 & 17 \end{vmatrix} \quad = \frac{1}{3}(-3)(17) = -17.$$

One can combine the use of row operations with expansion along a row or column:

12.4.2 Example Combine row operations with expansion along a row or column to find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 2 & 1 & 0 & 1 \\ 1 & 4 & -1 & 5 \\ 3 & 1 & 1 & 2 \end{bmatrix}.$$

Solution We have

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 1 & 2 & -1 & 6 \\ 2 & 1 & 0 & 1 \\ 1 & 4 & -1 & 5 \\ 3 & 1 & 1 & 2 \end{vmatrix} \begin{matrix} \\ \\ -1 \quad 1 \\ \\ \end{matrix} && \text{(make third column mostly zeros)} \\
&= \begin{vmatrix} 0 & -2 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 4 & -1 & 5 \\ 4 & 5 & 0 & 7 \end{vmatrix} && \text{(expand along third column)} \\
&= 0 + 0 + (-1)(+1) \begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 7 \end{vmatrix} + 0 \\
&= -\begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 7 \end{vmatrix} \begin{matrix} \\ -2 \\ \\ \end{matrix} && \text{(make first column mostly zeros)} \\
&= -\begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 5 \end{vmatrix} && \text{(expand along first column)} \\
&= -\left(0 + 2(-1) \begin{vmatrix} -2 & 1 \\ 3 & 5 \end{vmatrix} + 0 \right) \\
&= 2\big((-2)(5) - (1)(3)\big) = -26.
\end{aligned}$$

□

12.5 Cramer's rule

> CRAMER'S RULE.
>
> Let $\mathbf{A}$ be an $n \times n$ invertible matrix and let $\mathbf{b} \in \mathbf{R}^n$. Let $\mathbf{A}_i$ denote the matrix $\mathbf{A}$ with ith column replaced by $\mathbf{b}$. The equation $\mathbf{Ax} = \mathbf{b}$ has solution
>
> $$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}}$$
>
> $(i = 1, 2, \dots, n)$.

Proof. Since $\mathbf{A}^{-1}$ exists, we can solve the equation $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$:

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det \mathbf{A}}(\operatorname{Adj} \mathbf{A})\mathbf{b}, \end{aligned}$$

where we have used the theorem in Section 12.2. Comparing ith entries, we get

$$x_i = \frac{1}{\det \mathbf{A}}\left(c_{1i}b_1 + c_{2i}b_2 + \cdots c_{ni}b_n\right) = \frac{1}{\det \mathbf{A}} \det \mathbf{A}_i$$

the last equality being checked by expanding $\det \mathbf{A}_i$ along the ith column. □

12.5.1 Example Solve the following system using Cramer's rule:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8. \end{aligned}$$

Solution The corresponding matrix equation is $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix}.$$

We have

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{vmatrix} \\ &= 1(+1)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 3(-1)\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} + 1(+1)\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \\ &= (-3) - 3(0) + (6) = 3, \end{aligned}$$

so $\mathbf{A}$ is invertible and Cramer's rule applies. Next,

$$\begin{aligned}\det \mathbf{A}_1 &= \begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ -8 & 2 & -1 \end{vmatrix} \\ &= 1(+1)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 3(-1)\begin{vmatrix} 5 & 1 \\ -8 & -1 \end{vmatrix} + 1(+1)\begin{vmatrix} 5 & 1 \\ -8 & 2 \end{vmatrix} \\ &= (-3) - 3(3) + (18) = 6,\end{aligned}$$

so

$$x_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = \frac{6}{3} = 2.$$

Similarly,

$$\begin{aligned}\det \mathbf{A}_2 &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 1 \\ -2 & -8 & -1 \end{vmatrix} \\ &= 1(+1)\begin{vmatrix} 5 & 1 \\ -8 & -1 \end{vmatrix} + 1(-1)\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} + 1(+1)\begin{vmatrix} 2 & 5 \\ -2 & -8 \end{vmatrix} \\ &= (3) - (0) + (-6) = -3,\end{aligned}$$

so

$$x_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = \frac{-3}{3} = -1.$$

Finally,

$$\begin{aligned}\det \mathbf{A}_3 &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 5 \\ -2 & 2 & -8 \end{vmatrix} \\ &= 1(+1)\begin{vmatrix} 1 & 5 \\ 2 & -8 \end{vmatrix} + 3(-1)\begin{vmatrix} 2 & 5 \\ -2 & -8 \end{vmatrix} + 1(+1)\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \\ &= (-18) - 3(-6) + (6) = 6,\end{aligned}$$

so

$$x_3 = \frac{\det \mathbf{A}_3}{\det \mathbf{A}} = \frac{6}{3} = 2.$$

Therefore, the solution is $x_1 = 2$, $x_2 = -1$, and $x_3 = 2$. □

12–1 Find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{bmatrix}.$$

12–2 Find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & 3 \\ 5 & 1 & -2 & 0 \end{bmatrix}.$$

12–3 Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

by using the formula $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \operatorname{Adj} \mathbf{A}$ (see Section 12.2).

12–4 Let $\mathbf{A}$ be an invertible matrix. Show that $\det(\mathbf{A}^{-1}) = 1/(\det \mathbf{A})$.

HINT: Apply det to both sides of the equation $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

12–5 Use inspection to find the determinant of each of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 2 \\ 4 & -3 & -6 \end{bmatrix}$,

(b) $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$,

(c) $\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 2 & 7 \\ -1 & -1 & -1 & 0 \end{bmatrix}$.

12–6 Find the determinant of the following matrix by first applying row operations to obtain a triangular matrix

$$\mathbf{A} = \begin{bmatrix} 24 & 40 & -8 \\ 6 & 10 & 0 \\ 3 & 1 & 8 \end{bmatrix}.$$

12–7 Combine row operations with expansion along a row or column to find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & 4 & 2 \\ 1 & -1 & 1 & 5 \end{bmatrix}.$$

12–8 Solve the following system using Cramer's rule:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 \qquad + 4x_3 &= 2 \\ 3x_2 + x_3 &= 3. \end{aligned}$$