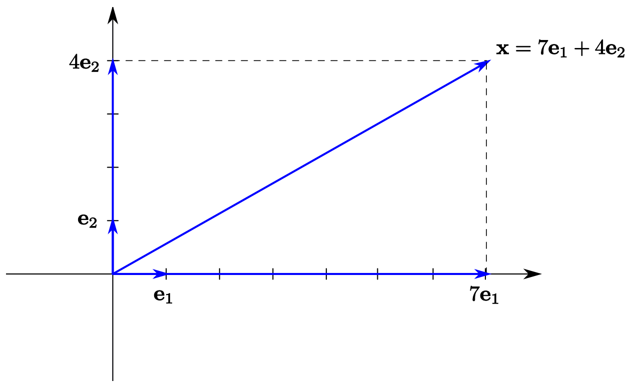
# 6 Basis

## 6.1 Introduction

If $\mathbf{x}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ are as pictured, then using the geometrical rules for scaling and adding vectors we see that $\mathbf{x} = 7\mathbf{e}_1 + 4\mathbf{e}_2$.

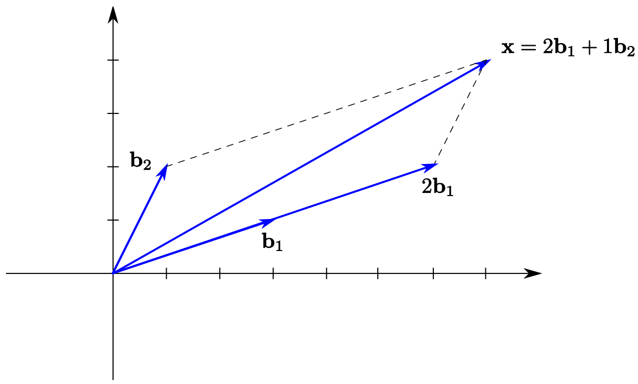



We say that $\mathbf{x}$ has $\mathbf{e}_1$-coordinate 7 and $\mathbf{e}_2$-coordinate 4. Similarly, we can find the coordinates of any vector with its tail at the origin by writing that vector as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$ and using the scalar factors. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ set up a coordinate system in the plane. (It is this coordinate system we use when we write $\mathbf{x} = [7, 4]^T$.)

Other vectors in the plane can be used to set up a coordinate system as well. For instance, let

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then $\mathbf{x} = 2\mathbf{b}_1 + 1\mathbf{b}_2$, so $\mathbf{x}$ has $\mathbf{b}_1$-coordinate 2 and $\mathbf{b}_2$-coordinate 1. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ also set up a coordinate system in the plane.



There are restrictions on which vectors can set up a coordinate system:

(i) The single vector $\mathbf{b}_1$ cannot set up a coordinate system, since, for instance, $\mathbf{x}$ cannot be written as a linear combination (= multiple) of $\mathbf{b}_1$, so there is no way to get the coordinates of $\mathbf{x}$.

(ii) The three vectors

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

cannot set up a coordinate system, since, for instance the vector $\mathbf{x} = [7, 4]^T$ can be written as a linear combination of these vectors in two different ways,

$$\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 + 5\mathbf{b}_3 \quad \text{and} \quad \mathbf{x} = 6\mathbf{b}_1 + (-1)\mathbf{b}_2 + 5\mathbf{b}_3,$$

and we cannot tell whether the coordinates of $\mathbf{x}$ should be $[-2, 3, 5]^T$ or $[6, -1, 5]^T$.

The problem in (i) is that the span of $\{\mathbf{b}_1\}$ is not all of $\mathbf{R}^2$. The problem in (ii) is that the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are not linearly independent. It turns out that if a collection of vectors spans the space we are trying to set up a coordinate system in and the vectors are linearly independent, then they can be used to set up a coordinate system. Such a collection is called a "basis" for the space.

## 6.2 Definition and examples

BASIS OF SUBSPACE.

Let $S$ be a subspace of $\mathbf{R}^n$ and let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ be vectors in $S$. The set $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\}$ is a **basis** for $S$ if

(i) $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s\} = S$.

(ii) $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ are linearly independent.

### 6.2.1 Example (Standard basis)

Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbf{R}^2$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

*Solution* Because of the wording "...is a basis for $\mathbf{R}^2$," the role of $S$ in the definition of basis is being played by $\mathbf{R}^2$. We check the two properties:

(i) (Span$\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbf{R}^2$?) The way to show that two sets are equal is to show that each is a subset of the other. Since linear combinations of vectors in $\mathbf{R}^2$ are still vectors in $\mathbf{R}^2$ we have Span$\{\mathbf{e}_1, \mathbf{e}_2\} \subseteq \mathbf{R}^2$. For the other inclusion, we let $\mathbf{x} = [x_1, x_2]^T$ be an arbitrary vector in $\mathbf{R}^2$. (Must show that $\mathbf{x}$ is in Span$\{\mathbf{e}_1, \mathbf{e}_2\}$.) We need to show that $\mathbf{x} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ for some scalars $\alpha_1$ and $\alpha_2$, that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can let $\alpha_1 = x_1$ and $\alpha_2 = x_2$. (We used inspection here, but usually one has to solve a system.) Therefore, $\mathbf{R}^2 \subseteq$ Span$\{\mathbf{e}_1, \mathbf{e}_2\}$ and, since we showed the other inclusion earlier, we have Span$\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbf{R}^2$.

(ii) ($\mathbf{e}_1, \mathbf{e}_2$ linearly independent?) Neither vector is a linear combination (here, multiple) of the other, so $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent.

Therefore, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbf{R}^2$. □

More generally, the set of standard unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$, called the **standard basis**.

**6.2.2 Example** Show that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$, where

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

*Solution* We check the two properties.

(i) (Span$\{\mathbf{b}_1, \mathbf{b}_2\} = \mathbf{R}^2$?) Every linear combination of vectors in $\mathbf{R}^2$ is a vector in $\mathbf{R}^2$ so Span$\{\mathbf{b}_1, \mathbf{b}_2\} \subseteq \mathbf{R}^2$. Let $\mathbf{x} = [x_1, x_2]^T$ be an arbitrary vector in $\mathbf{R}^2$. (Must show that $\mathbf{x}$ is in Span$\{\mathbf{b}_1, \mathbf{b}_2\}$.) We need to show that $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$ for some scalars $\alpha_1$ and $\alpha_2$, that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\alpha_1 + \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix}.$$

This leads to a system with corresponding augmented matrix

$$\left[\begin{array}{cc|c} 3 & 1 & x_1 \\ 1 & 2 & x_2 \end{array}\right] \begin{matrix} \\ -3 \end{matrix} \quad \sim \quad \left[\begin{array}{cc|c} 3 & 1 & x_1 \\ 0 & -5 & x_1 - 3x_2 \end{array}\right].$$

Since there is no pivot in the augmented column, a solution exists. Therefore, $\mathbf{R}^2 \subseteq$ Span$\{\mathbf{b}_1, \mathbf{b}_2\}$, and, since we showed the other inclusion earlier, we have Span$\{\mathbf{b}_1, \mathbf{b}_2\} = \mathbf{R}^2$.

(ii) ($\mathbf{b}_1, \mathbf{b}_2$ linearly independent?) Neither vector is a linear combination (here, multiple) of the other, so $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent.

Therefore, $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$. □

**6.2.3 Example** Show that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $S = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

*Solution* We check the two properties of basis:

(i) ($\text{Span}\{\mathbf{b}_1, \mathbf{b}_2\} = S$?) Since $S$ is *defined* to be $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$ this property is automatically satisfied.

(ii) ($\mathbf{b}_1, \mathbf{b}_2$ linearly independent?) Neither vector is a linear combination (here, multiple) of the other, so $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent.

Therefore, $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $S$. □

**6.2.4 Example** Determine whether $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbf{R}^3$ (where $\mathbf{e}_1 = [1, 0, 0]^T$, $\mathbf{e}_2 = [0, 1, 0]^T$).

*Solution* $\text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$ consists of all vectors of the form $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 = [\alpha_1, \alpha_2, 0]^T$ (the third component must be zero). Since $[0, 0, 1]^T$ is a vector in $\mathbf{R}^3$ that is not in $\text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$ we see that $\text{Span}\{\mathbf{e}_1, \mathbf{e}_2\} \neq \mathbf{R}^3$. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is *not* a basis for $\mathbf{R}^3$.

(This counterexample was obtained by inspection. Here is some possible scratch work. Let $\mathbf{x} = [x_1, x_2, x_3]^T$ be an arbitrary vector in $\mathbf{R}^3$. Trying to write $\mathbf{x}$ as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$ leads to a system with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{array}\right].$$

We see that there is no solution if $x_3 \neq 0$. This gives the idea for the counterexample above.) □

**6.2.5 Example** Determine whether $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $S = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 6 \\ 0 \\ -3 \\ 9 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

*Solution* By inspection, the second vector is a multiple of the third ($\mathbf{b}_2 = 3\mathbf{b}_3$), so the vectors are not linearly independent. Therefore, the set is not a basis for $S$.

(If inspection such as this had been difficult, we would instead have applied row operations to the matrix having the vectors as columns, namely, $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix}$. We would have found that this matrix does not have full rank, so the vectors are not linearly independent by the theorem before Example 5.2.3.) □

## 6.3 Coordinate vector

Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the vectors in $\mathbf{R}^2$ given by

$$\mathbf{b}_1 = \begin{bmatrix}3\\1\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}.$$

Since $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$ (see Example 6.2.2), any vector in $\mathbf{R}^2$ can be written *uniquely* as a linear combination of these two vectors (a vector can be written at least one way as a linear combination due to the spanning property of basis and at most one way due to the linear independence property of basis). For instance, if $\mathbf{x} = [7, 4]^T$, then

$$\mathbf{x} = 2\mathbf{b}_1 + 1\mathbf{b}_2.$$

We say that the "coordinate vector" of $\mathbf{x}$ relative to the ordered basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ is $[2, 1]^T$. In symbols,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\1\end{bmatrix}.$$

This definition requires a specific ordering of the basis since, if the order is not fixed, then someone else might write $\mathbf{x} = 1\mathbf{b}_2 + 2\mathbf{b}_1$ and say that the coordinate vector of $\mathbf{x}$ is $[1, 2]^T$.

Here is the general definition:

Coordinate vector.

Let $S$ be a subspace of $\mathbf{R}^n$, let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s)$ be an ordered basis for $S$, and let $\mathbf{x}$ be a vector in $S$. The **coordinate vector of x relative to $\mathcal{B}$** is

$$[\mathbf{x}]_{\mathcal{B}} = [\alpha_1, \alpha_2, \ldots, \alpha_s]^T,$$

where $\mathbf{x} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_s\mathbf{b}_s$.

Put more simply, the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ is found by writing $\mathbf{x}$ as a linear combination of the (ordered) basis vectors, pulling off the scalar factors, and writing those factors as a column matrix.

**6.3.1 Example** Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be the ordered basis of $\mathbf{R}^2$ with

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = [9, 8]^T$.

(b) Sketch the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{x}$, as well as the coordinate system grid determined by $\mathcal{B}$.

*Solution* (a) We need to write $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$:

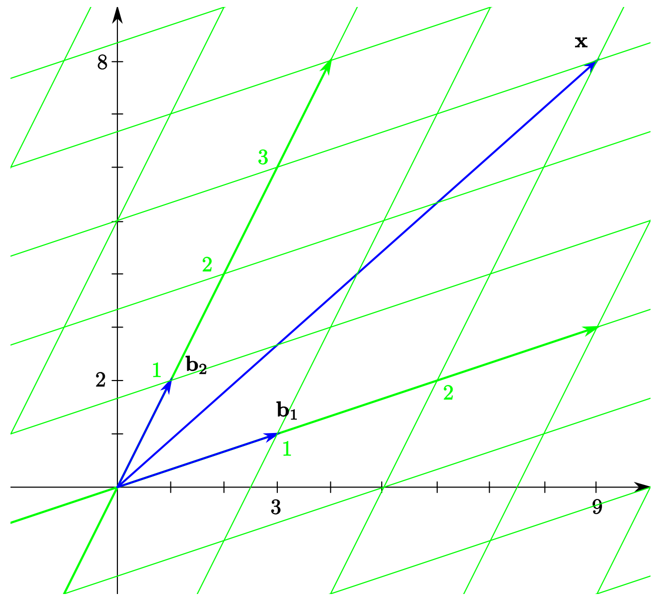
$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$$
$$\begin{bmatrix} 9 \\ 8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\alpha_1 + \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix}.$$

Equating components leads to a system with augmented matrix

$$\left[\begin{array}{cc|c} 3 & 1 & 9 \\ 1 & 2 & 8 \end{array}\right]_{-3} \sim \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 0 & -5 & -15 \end{array}\right]_{-\frac{1}{5}}$$
$$\sim \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 0 & 1 & 3 \end{array}\right]_{-1}$$
$$\sim \left[\begin{array}{cc|c} 3 & 0 & 6 \\ 0 & 1 & 3 \end{array}\right]^{\frac{1}{3}}$$
$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Therefore, $\mathbf{x} = 2\mathbf{b}_1 + 3\mathbf{b}_2$ so that $[\mathbf{x}]_{\mathcal{B}} = [2, 3]^T$.

(b) Here is the sketch:




The ordered basis $\mathcal{B}$ sets up a new coordinate system (with grid in green). In this coordinate system the vector $\mathbf{x}$ has $\mathbf{b}_1$-coordinate 2 and $\mathbf{b}_2$-coordinate 3.

□

**6.3.2 Example** Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be the ordered basis of Span$\{\mathbf{b}_1, \mathbf{b}_2\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = [1, 4, 3]^T$.

(b) Find $\mathbf{y}$, given that $[\mathbf{y}]_{\mathcal{B}} = [9, -2]^T$.

*Solution* (a) We need to write $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$$
$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \\ \alpha_2 \end{bmatrix}.$$

Equating components shows immediately that $\alpha_1 = 1$ and $\alpha_2 = 3$. (In this case, there is no need to use the augmented matrix of the system.) Therefore, $\mathbf{x} = 1\mathbf{b}_1 + 3\mathbf{b}_2$ so that $[\mathbf{x}]_{\mathcal{B}} = [1, 3]^T$.

(b) We are given that $[\mathbf{y}]_{\mathcal{B}} = [9, -2]^T$. Therefore,

$$\mathbf{y} = 9\mathbf{b}_1 + (-2)\mathbf{b}_2 = 9\begin{bmatrix}1\\1\\0\end{bmatrix} + (-2)\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}9\\7\\-2\end{bmatrix}.$$

□

**6.3.3 Example** Let $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard ordered basis of $\mathbf{R}^3$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{E}}$, where $\mathbf{x} = [4, 3, -5]^T$.

*Solution* Using inspection, we have

$$\mathbf{x} = \begin{bmatrix}4\\3\\-5\end{bmatrix} = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + (-5)\begin{bmatrix}0\\0\\1\end{bmatrix} = 4\mathbf{e}_1 + 3\mathbf{e}_2 + (-5)\mathbf{e}_3.$$

Therefore, $[\mathbf{x}]_{\mathcal{E}} = [4, 3, -5]^T$. □

The example illustrates the fact that if $\mathcal{E}$ is the standard basis for $\mathbf{R}^n$, then

$$[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$$

for every vector $\mathbf{x}$ in $\mathbf{R}^n$.

Put another way, when we specify a vector by giving a list of numbers, like $\mathbf{x} = [4, 3, -5]^T$, we are really giving the vector's coordinate vector relative to the standard basis.

## 6 – Exercises

**6–1** Show that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbf{R}^2$, where

$$\mathbf{b}_1 = \begin{bmatrix}2\\2\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix}-4\\2\end{bmatrix}.$$

**6–2** Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbf{R}^3$, where

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -1 \\ -5 \\ 5 \end{bmatrix}.$$

**6–3** Determine whether $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $S = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}.$$

**6–4** Determine whether $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $S = \{\mathbf{x} \in \mathbf{R}^3 \mid 3x_1 + x_2 + 4x_3 = 0\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}.$$

**6–5** Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be the ordered basis of $\mathbf{R}^2$ with

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

(a) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = [-1, 5]^T$.

(b) Sketch the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{x}$, as well as the coordinate system grid determined by $\mathcal{B}$.

**6–6** Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be the ordered basis of $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = [-1, 7, 7]^T$.

(b) Find $\mathbf{y}$, given that $[\mathbf{y}]_{\mathcal{B}} = [4, -6]^T$.