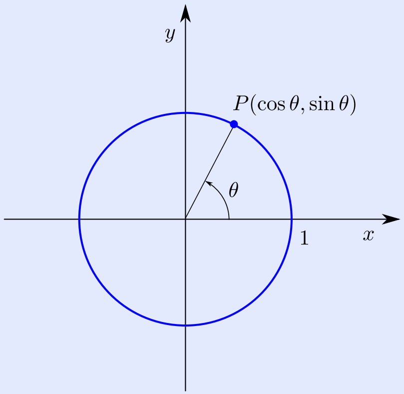
4. Trigonometry

4.1. Basic concepts

Sine and cosine. A given angle θ determines the indicated point P on the unit circle. The convention is to measure the angle θ from the positive x-axis going counterclockwise if $\theta > 0$; clockwise if $\theta < 0$.



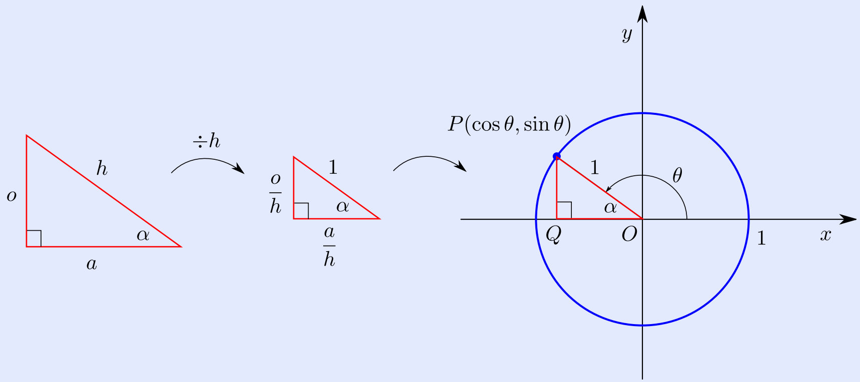
The **cosine** of θ (written $\cos\theta$) is defined to be the x-coordinate of this point; the **sine** of θ (written $\sin\theta$) is defined to be the y-coordinate of this point.

The sine or cosine of a multiple of 90° is easy to find since the corresponding point P lies on one of the coordinate axes. For example,

$$\begin{aligned}
\cos 90^\circ &= 0 && (x\text{-coordinate of the point } P(0,1)),\\
\sin 90^\circ &= 1 && (y\text{-coordinate of the point } P(0,1)),\\
\cos 180^\circ &= -1 && (x\text{-coordinate of the point } P(-1,0)),\\
\sin 180^\circ &= 0 && (y\text{-coordinate of the point } P(-1,0)),
\end{aligned}$$

and so on.

For the sine or cosine of an angle that is not a multiple of 90°, there is a method that can sometimes be used: Let θ be as pictured below and suppose we want to find $\cos\theta$ and $\sin\theta$. What we need are the coordinates of the point P; we could figure these out if we knew the lengths of the legs of the right triangle $\triangle OPQ$.

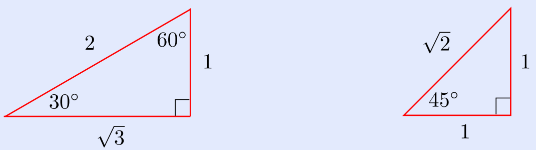


Suppose we knew the leg lengths of some triangle similar to $\triangle OPQ$. Such a triangle is pictured with side lengths labeled a (adjacent to α), o (opposite α), and h (hypotenuse). By dividing all lengths by h we can scale our triangle so that it has hypotenuse 1 and is therefore exactly the same size as the triangle $\triangle OPQ$ (which we know has hypotenuse of length 1 since the circle has radius 1). It follows that

$$\cos\theta = \pm\frac{a}{h} \qquad \text{and} \qquad \sin\theta = \pm\frac{o}{h}, \tag{1}$$

where the signs are determined by the quadrant. (In the diagram, since P is in the second quadrant, its x-coordinate is negative, so that $\cos\theta = -a/h$, and its y-coordinate is positive, so that $\sin\theta = o/h$.)

The described method requires that we know the side lengths of some right triangle with angle α (called the **reference angle**). Here are two triangles that can be used for computing the cosine and sine of multiples of 30° and multiples of 45°:



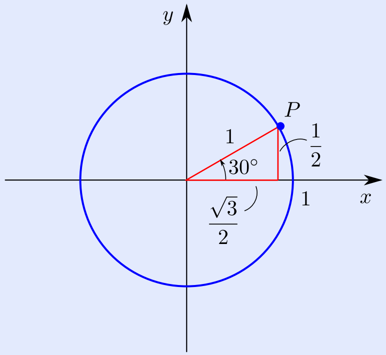
For the first triangle, the length of the short side was arbitrarily chosen to be 1; the length of the hypotenuse is then forced to be 2 (this is half of an equilateral triangle), and the length of the remaining leg is forced to be $\sqrt{3}$ by the Pythagorean theorem. For the second triangle, after the length of one leg is chosen to be 1, the length of the other leg is forced to be 1 as well (this is an isosceles triangle); the length of the hypotenuse is then forced to be $\sqrt{2}$ by the Pythagorean theorem again.

4.1.1 Example

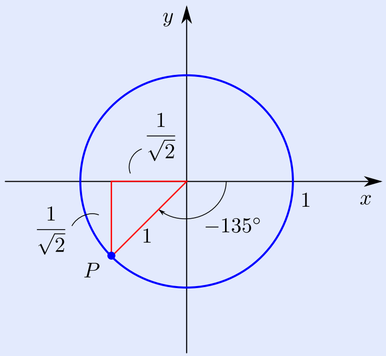
(a) Find $\cos 30°$.

(b) Find $\sin(-135°)$.

Solution (a) Scaling the 30-60-90 triangle down by a factor of 2 (in order to make the hypotenuse have length 1), we get the indicated triangle. Since $\cos 30°$ is the x-coordinate of P, we conclude that $\cos 30° = \sqrt{3}/2$.



(b) Scaling the 45° right triangle down by a factor of $\sqrt{2}$, we get the indicated triangle. Since $\sin(-135°)$ is the y-coordinate of P, we conclude that $\sin(-135°) = -1/\sqrt{2}$ (the negative sign coming from inspection).



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We will not attempt to find sines or cosines of angles that are not multiples of either 30° or 45°. For instance, if $\cos 34°$ appears in a problem, we will just leave it like this; or, if the problem asks for an approximation, we will use a calculator to get $\cos 34° \approx 0.83$.

4.1.2 Example A right triangle has angle 34° with adjacent leg of length 4. Find the length of the hypotenuse. (Give an exact answer as well as an approximation.)

Solution Using (1) (with positive sign since the point P corresponding to 34° is in the first quadrant), we get

$$\cos 34° = \frac{a}{h} = \frac{4}{h},$$

so the length of the hypotenuse is (exactly) $h = 4/\cos 34°$, or approximately $4/0.83 = 4.82$.

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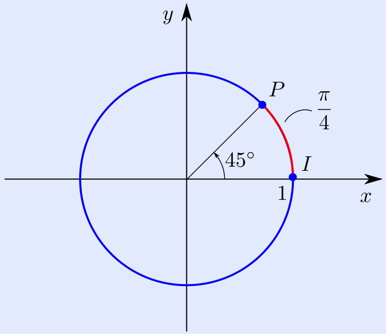
Degree versus radian. The angle 45° determines the pictured point P on the unit circle and hence the arc $\overset{\frown}{IP}$.



The length of this arc is one-eighth the circumference of the unit circle, so

$$\text{length}\,\overset{\frown}{IP} = \frac{1}{8}(2\pi r) = \frac{1}{8}(2\pi(1)) = \frac{\pi}{4}.$$

Conversely, if we specify that the point P is to be located by starting at the point I and moving $\pi/4$ units along the unit circle (counterclockwise), then this determines the angle 45°.

This relationship between degree measure and arc length is expressed by writing $45° = \pi/4\,\text{rad}$ (rad is for **radian**). The illustrated method generalizes to allow any angle to be measured in radians instead of degrees. Here are some commonly occurring degree measures and their radian equivalents:

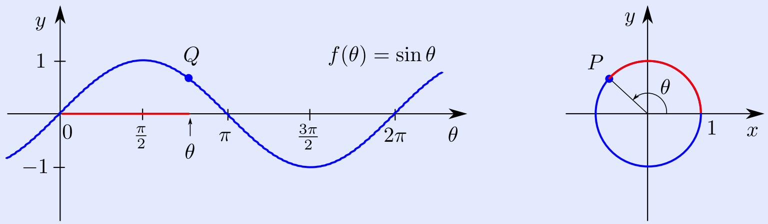
$$360^\circ = 2\pi \text{ rad} \qquad 180^\circ = \pi \text{ rad} \qquad 90^\circ = \frac{\pi}{2} \text{ rad}$$

$$60^\circ = \frac{\pi}{3} \text{ rad} \qquad 45^\circ = \frac{\pi}{4} \text{ rad} \qquad 30^\circ = \frac{\pi}{6} \text{ rad}$$

In calculus, radian measure is preferred over degree measure since a number of formulas can be expressed in terms of radians more simply than in terms of degrees. Because of this, our default unit will be the radian and we will generally suppress the designation "rad." For instance, we will write $\sin(\pi/2) = 1$.

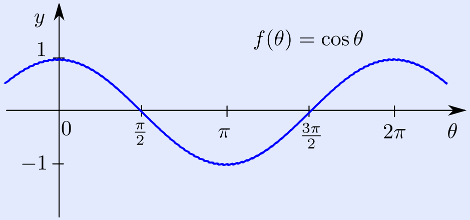
4.2. Trigonometric functions

The **sine function** $f(\theta) = \sin\theta$ is obtained by viewing θ as an input value and $\sin\theta$ as the corresponding output value. Here is the graph:



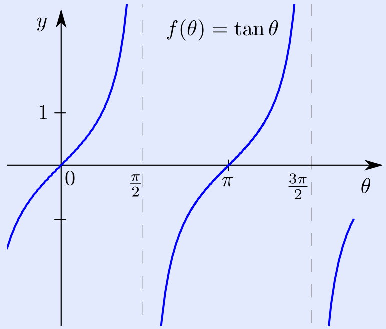
The unit circle has been reproduced to the right in order to show the connection between the graph of the function and the definition of $\sin\theta$: The angle is measured in radians, so the red arc on the unit circle has length θ as does the red line segment on the θ-axis. The points P and Q both have the same height, namely $\sin\theta$. The graph repeats every 2π units.

The graph of the **cosine function**, defined by $f(\theta) = \cos\theta$, is given below:

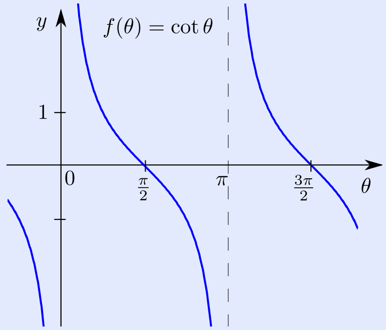


The **tangent**, **cotangent**, **secant**, and **cosecant functions** are defined in terms of $\sin\theta$ and $\cos\theta$:

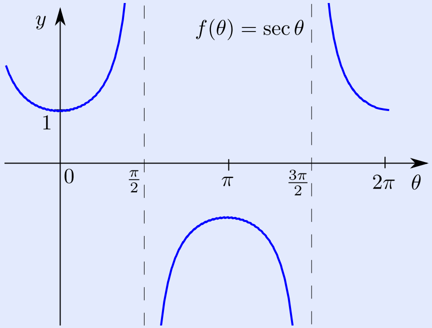
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$



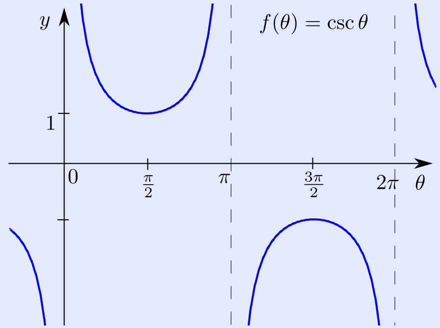
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$



$$\sec\theta = \frac{1}{\cos\theta}$$

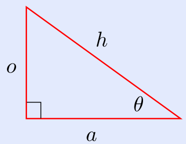


$$\csc\theta = \frac{1}{\sin\theta}$$



The six functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ are the **trigonometric functions**.

Here are formulas for computing the values of these trigonometric functions when θ is acute and appears as one of the angles of a known right triangle:



$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a} \qquad \cot\theta = \frac{a}{o}$$

$$\sec\theta = \frac{h}{a} \qquad \csc\theta = \frac{h}{o}$$

Table of Contents
Page 10 of 16
Back
Print Version
Home Page

The formulas for $\sin\theta$ and $\cos\theta$ follow from (1) of 4.1. We have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{o/h}{a/h} = \frac{o}{a},$$

which gives the next formula. The remaining formulas are derived similarly.

4.3. Trigonometric identities

An angle θ corresponds to a point P on the unit circle and this point has coordinates $(\cos\theta, \sin\theta)$ (see 4.1). The unit circle has equation $x^2 + y^2 = 1$ (due to the Pythagorean theorem), so the coordinates of P must satisfy this equation, giving

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

This is the most frequently used trigonometric identity. For reference, we provide a list of some trigonometric identities used in calculus:

$$\sin^2\theta + \cos^2\theta = 1, \tag{1}$$

$$\tan^2\theta + 1 = \sec^2\theta, \tag{2}$$

$$1 + \cot^2\theta = \csc^2\theta, \tag{3}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{4}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \tag{5}$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta), \tag{6}$$

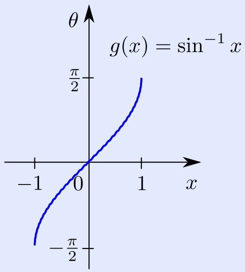
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta). \tag{7}$$

The identity (1) is just the identity derived above with $(\sin\theta)^2$ written $\sin^2\theta$ and $(\cos\theta)^2$ written $\cos^2\theta$ as is customary (but see word of caution below). The identities (2) and (3) are derived by dividing (1) by $\cos^2\theta$ and $\sin^2\theta$, respectively. Derivations of the identities (4) and (5) require a little more work, but depend ultimately on the Pythagorean theorem. Finally, the identities (6) and (7) follow from the preceding identities (see Exercise 4−3).

4.4. Inverse trigonometric functions

Let $f(\theta) = \sin\theta$. We would like an inverse of this function, that is, a function that sends an output of this function back to the original input. Unfortunately, since f is not injective (its graph fails the horizontal line test), such an inverse does not exist. However, just as we did with the squaring function (see 2.5), we can get around this problem by reducing the domain of f.

The function $f(\theta) = \sin\theta$ with domain $[-\pi/2, \pi/2]$ is injective and therefore it has an inverse, which we denote $g(x) = \sin^{-1} x$. This function is called the **inverse sine function** (or **arc sine function**). Its domain is $[-1, 1]$, its range is $[-\pi/2, \pi/2]$, and its graph is obtained by reflecting the graph of f across the 45° line $x = \theta$:



4.4.1 Example

(a) Find $\sin^{-1}(-\sqrt{3}/2)$.

(b) Solve the equation $\sin\theta = -\sqrt{3}/2$.

Solution

(a) We are trying to find a certain angle θ, namely the θ in the interval $[-\pi/2, \pi/2]$ such that $\sin\theta = -\sqrt{3}/2$. Such a θ corresponds to a point P on the unit circle with y-coordinate $-\sqrt{3}/2$. Because of the negative sign, we know at least that P is below the x-axis, so θ must be negative. Saying any more than this is often not possible (and we resort to using a calculator if an approximation is called for).

However, recalling the 30-60-90 triangle (see 4.1) we recognize the ratio $\sqrt{3}/2$ as being the ratio o/h, where o is the length of the leg opposite 60° and h is the hypotenuse. Therefore, the reference angle corresponding to P is $60^\circ = \pi/3$, and we conclude that $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.

Table of Contents
Page 13 of 16
Back
Print Version
Home Page

(b) We are to find all values of θ that satisfy the equation. In part (a) we found that the angle $\theta = -\pi/3$, corresponding to a point P in the fourth quadrant with y-coordinate $-\sqrt{3}/2$, satisfies the given equation. A multiple of 2π added to this angle yields an angle that also corresponds to the point P and therefore satisfies the equation. Mentally traversing the unit circle, looking for any other points with y-coordinate $-\sqrt{3}/2$, we find that there is exactly one; it is in the third quadrant and it corresponds to the angle $-2\pi/3$, or any multiple of 2π added to this. The answer is $\theta = -\pi/3 + 2\pi n, -2\pi/3 + 2\pi n$, where n is any integer.

□

The previous example is analogous to Example 2.5.1.

The other trigonometric functions have inverses as well, provided we reduce their domains in order to achieve injective functions. The domains and ranges of these inverse functions are given below.

	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

The sets listed in the range column are the reduced domains of the corresponding trigonometric functions. The graph of each inverse trigonometric function is the reflection across the 45° line $x = \theta$ of the portion of the graph of the corresponding trigonometric function that is left after reducing the domain.

Word of caution about notation. Generally, $\sin^n \theta$ is taken to mean $(\sin \theta)^n$. The one exception is the case $n = -1$:

$$\sin^{-1} x \neq (\sin x)^{-1},$$

that is, $\sin^{-1} x$ is *not* the reciprocal of $\sin x$. Rather, $\sin^{-1} x$ always represents the inverse sine function (and similarly for the other inverse trigonometric functions). This notation is used because of the convention that, if f is the name of a function, then f^{-1} is the name of its inverse (see 2.5): sin is the name of the sine function, so $\sin^{-1}$ should be the name of its inverse.

Some people write $\arcsin x$ instead of $\sin^{-1} x$ in order to avoid any confusion, but the notation $\sin^{-1} x$ is so common that it is a good idea to learn how to use it (with care).

4 – Exercises

4 – 1 Find the following without using a calculator. (Sketch the corresponding point P on the unit circle as well as the triangle you use to work out the answer, if appropriate):

(a) $\cos 135°$

(b) $\sin(4\pi/3)$

(c) $\sin(-3\pi/2)$

(d) $\tan(\pi/6)$

4 – 2 Sketch the graph of the equation $y = 1 - \sin(\theta - \pi/4)$.

HINT: Use the method of Exercise 3 – 2.

4 – 3 Derive the trigonometric identity $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ using only the identities (1)-(5) of 4.3.

4 – 4

(a) Find $\cos^{-1}(-1/\sqrt{2})$.

(b) Solve the equation $\cos \theta = -1/\sqrt{2}$.