27. Related rates

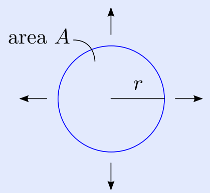
27.1. Method

When one quantity depends on a second quantity, any change in the second quantity effects a change in the first and the rates at which the two quantities change are related. The study of this situation is the focus of this section.

A rate of change is given by a derivative: If $y = f(t)$, then $\frac{dy}{dt}$ (meaning the derivative of y) gives the (instantaneous) rate at which y is changing with respect to t (see 14).

27.1.1 Example The radius of a circle is increasing at a constant rate of 2 cm/s. Find the rate at which the area of the circle is changing when the radius is 5 cm.

Solution Let r denote the radius of the circle and let A denote the circle's area.



The given information and the quantity to be found, expressed using symbols, are as follows:

- Given: $\dfrac{dr}{dt} = 2$
- Want: $\left.\dfrac{dA}{dt}\right|_{r=5}$

(As long as the units are consistent, there is no need to continually write them. Our policy is to convert to consistent units at the outset of the solution, if necessary, and then suppress the units until the end.)

The relationship between the area A and the radius r is as follows:

- Relationship: $A = \pi r^2$

Implicit differentiation of this equation with respect to time t gives

$$\frac{d}{dt}[A] = \frac{d}{dt}\left[\pi r^2\right]$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r(2) = 4\pi r,$$

where we have used the given information.

Evaluating at $r = 5$ gives the answer:

$$\left.\frac{dA}{dt}\right|_{r=5} = 4\pi(5) = 20\pi \text{ cm}^2/\text{s}$$

(or approximately 62.8 cm^2/s). □

The example illustrates the steps one typically takes in solving a related rates problem.

SOLVING A RELATED RATES PROBLEM.

(i) Sketch a diagram showing the ongoing situation and label relevant quantities.

(ii) Express the given information and the quantity to be found using symbols.

(iii) Write an equation expressing the relationship between the quantities.

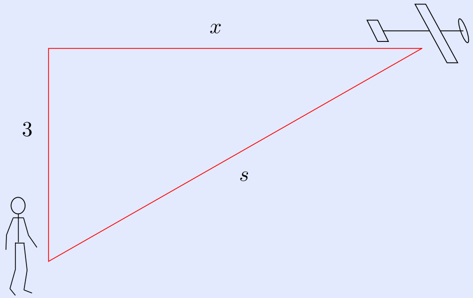
(iv) Use implicit differentiation to find the desired derivative.

(v) If required, evaluate the derivative at the specified value(s).

27.2. Examples

27.2.1 Example An airplane flying horizontally at an altitude of 3000 m and a speed of 480 k/hr passes directly above an observer on the ground. How fast is the distance from the observer to the airplane increasing 30 s later?

Solution We begin by sketching a diagram:



The units in the statement of the problem are mixed. In the diagram, we have converted the 3000 m altitude to 3 k. Converting 30 s to 1/120 hr below leaves us with consistent units (kilometers and hours), and we can safely forget about units until the end.

- Given: $\dfrac{dx}{dt} = 480$
- Want: $\left.\dfrac{ds}{dt}\right|_{t=1/120}$

The relationship between the variables comes from the Pythagorean theorem:

- Relationship: $s^2 = 3^2 + x^2$

Differentiating implicitly with respect to t, we get

$$\begin{aligned} \frac{d}{dt}\left[s^2\right] &= \frac{d}{dt}\left[3^2 + x^2\right] \\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} \\ \frac{ds}{dt} &= \frac{480x}{s}, \end{aligned}$$

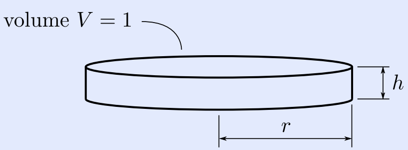
where we have used the given information. The final step is to evaluate ds/dt at $t = 1/120$. The formula we obtained requires that we find x and s corresponding to this particular time. The plane, moving at a constant rate of 480 k/hr, travels 4 k in 1/120 hr, so $x = 4$. The corresponding s is 5 as can be seen from the relationship. Therefore,

$$\left.\frac{ds}{dt}\right|_{t=1/120} = \left.\frac{ds}{dt}\right|_{\substack{x=4\\ s=5}} = \frac{(480)(4)}{5} = 384 \text{ k/hr}.$$

□

27.2.2 Example A circular oil slick of uniform thickness is caused by a spill of 1 m^3 of oil. The thickness of the oil slick is decreasing at a rate of 0.1 cm/hr. At what rate is the radius of the slick increasing when it is 8 m?

Solution The oil slick has the shape of a cylinder:



After converting 0.1 cm/hr to 0.001 m/hr, we have

- Given: $V = 1, \quad \frac{dh}{dt} = -0.001$
- Want: $\left.\frac{dr}{dt}\right|_{r=8}$

(The negative sign is required since the thickness of the oil slick is decreasing with time.) The volume of a cylinder is the area of its base times its height:

- Relationship: $1 = V = \pi r^2 h$

Using implicit differentiation, we have

$$\begin{aligned} \frac{d}{dt}[1] &= \frac{d}{dt}\left[\pi r^2 h\right] \\ 0 &= \frac{d}{dt}\left[\pi r^2\right] h + \pi r^2 \frac{d}{dt}[h] \\ 0 &= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}, \end{aligned}$$

so

$$\frac{dr}{dt} = -\frac{r}{2h} \cdot \frac{dh}{dt} = \frac{0.0005r}{h},$$

the last step using the given information. Before we evaluate dr/dt at $r = 8$, we use the relationship to find that the corresponding thickness of the oil slick is $h = 1/(64\pi)$.

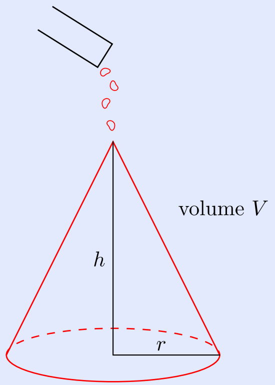
Therefore,

$$\left.\frac{dr}{dt}\right|_{r=8} = \left.\frac{dr}{dt}\right|_{\substack{r=8\\h=1/(64\pi)}} = \frac{(0.0005)(8)}{1/(64\pi)} = 0.256\pi \text{ m/hr}$$

(or approximately 0.804 m/hr). □

27.2.3 Example Soybeans, pouring down from a chute at a constant rate of 2 m^3/min, form a conical hill. Assuming that the height of the hill is always twice the radius of its base, find the rate at which the height is increasing at the moment the height is 1 m, and also when the height is 4 m.

Solution The hill of beans looks like this:



The units are already consistent, so we suppress them until the end.

- Given: $\dfrac{dV}{dt} = 2, \qquad h = 2r$
- Want: $\left.\dfrac{dh}{dt}\right|_{h=1}$ and $\left.\dfrac{dh}{dt}\right|_{h=4}$

The relationship is from the formula for the volume of a cone:

- Relationship: $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$

(The given information $h = 2r$ has been used to get the final expression.) Using implicit differentiation, we get

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}\left[\tfrac{1}{12}\pi h^3\right] \\ \frac{dV}{dt} &= \tfrac{1}{4}\pi h^2 \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{8}{\pi h^2}, \end{aligned}$$

the last step using the given information. The two desired rates are

$$\left.\frac{dh}{dt}\right|_{h=1} = 8/\pi \text{ m/min} \approx 2.5 \text{ m/min}$$

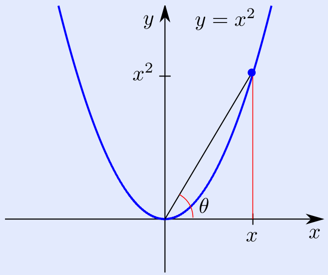
and

$$\left.\frac{dh}{dt}\right|_{h=4} = 1/2\pi \text{ m/min} \approx 0.2 \text{ m/min}.$$

□

27.2.4 Example A particle is moving along the graph of $y = x^2$ in such a way that its x-coordinate is increasing at a constant rate of 10 units per second. Let θ be the angle between the positive x-axis and the line joining the particle to the origin. Find the rate at which θ is changing when the x-coordinate of the point is 3.

Solution The following graph shows the ongoing situation:



The units are already consistent, so we suppress them until the end.

- Given: $\dfrac{dx}{dt} = 10$
- Want: $\left.\dfrac{d\theta}{dt}\right|_{x=3}$

Using the indicated triangle in the diagram, we get the following relationship between θ and x:

- Relationship: $\tan\theta = \dfrac{x^2}{x} = x$

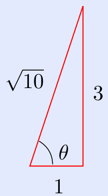
Differentiating this equation implicitly with respect to t, we get

$$\frac{d}{dt}[\tan\theta] = \frac{d}{dt}[x]$$
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt},$$

so

$$\frac{d\theta}{dt} = 10\cos^2\theta,$$

where we have used the given information. In order to evaluate this last equation when $x = 3$ we need to find the corresponding angle θ. Actually, it is sufficient just to find the cosine of this corresponding angle. The relationship gives $\tan\theta = 3$, so, in a right triangle with one angle equal to θ, we have $o/a = 3/1$, for instance



The Pythagorean theorem says that the length of the hypotenuse is $\sqrt{10}$, so $\cos\theta = a/h = 1/\sqrt{10}$.

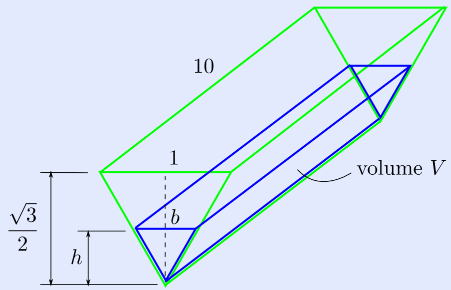
Therefore,

$$\left.\frac{d\theta}{dt}\right|_{x=3} = 10\left(\frac{1}{\sqrt{10}}\right)^2 = 1 \text{ rad/s.}$$

□

27.2.5 Example A water trough of length 10 m has cross section an equilateral triangle of side 1 m. The trough is being filled at a constant rate of 2 m^3/min. Find the rate at which the level of the water in the trough is rising when the water is 50 cm deep.

Solution The following diagram shows the ongoing situation:



For later use, we have recorded the height of the trough, which can be determined either from the memorized 30-60-90 triangle or from the Pythagorean theorem. We get consistent units by converting 50 cm to 0.5 m, and so we suppress the units until the end.

- Given: $\dfrac{dV}{dt} = 2$
- Want: $\left.\dfrac{dh}{dt}\right|_{h=0.5}$

The relationship comes from the formula for the volume of a triangular prism:

- Relationship: $V = (\text{area of end}) \cdot (\text{length}) = \left(\frac{1}{2}bh\right)(10) = 5bh.$

The similar triangles at the end of the trough relate b to h:

$$\frac{b}{h} = \frac{1}{\sqrt{3}/2},$$

so that

$$b = \frac{2h}{\sqrt{3}}.$$

Substitution of this expression for b into the relationship gives

$$V = \frac{10}{\sqrt{3}}h^2.$$

We differentiate this last equation implicitly with respect to t:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{10}{\sqrt{3}}h^2\right]$$
$$\frac{dV}{dt} = \frac{20}{\sqrt{3}}h\frac{dh}{dt},$$

so

$$\frac{dh}{dt} = \frac{\sqrt{3}}{20h}\frac{dV}{dt} = \frac{\sqrt{3}}{10h},$$

where we have used the given information. Evaluation at $h = 0.5$ gives the answer:

$$\left.\frac{dh}{dt}\right|_{h=0.5} = \frac{\sqrt{3}}{10(0.5)} = \frac{\sqrt{3}}{5} \text{ m/min} \approx 35 \text{ cm/min}.$$

□

27 – Exercises

27 – 1 A meteor (spherical in shape) enters earth's atmosphere and starts burning up in such a way that its surface area decreases at a constant rate of 100 cm^2/s. Find the rate at which the diameter is changing when the radius is 5 m.

HINT: The surface area of a sphere of radius r is $4\pi r^2$.

27 – 2 The width of a rectangle is increasing at a rate of 3 cm/s while its height is decreasing at a rate of 4 cm/s. Find the rate at which the area of the rectangle is changing when the width is 12 cm and the height is 20 cm.

27 – 3 A particle moves on the line $y = x + 2$ in such a way that its x-coordinate changes at the constant rate of 5 units per second. A right triangle is formed by the line joining the particle to the origin, the vertical line from the particle to the x-axis, and the x-axis. Find the rate at which the area of this triangle is changing when the particle is at the point $(3, 5)$.

27 – 4 Zoe is flying a kite. The kite is initially directly over her head at a height of 30 m. Then the wind starts carrying the kite in one direction at a rate of 40 m/min while Zoe starts running in the opposite direction at a rate of 200 m/min. Assuming that the height of the kite remains constant and that the string forms a straight line, find the rate at which the string is paying out after 10 s.

27 – 5 The ice cream in a sugar cone (conically shaped) has all melted and is dripping out a hole in the bottom of the cone at a constant rate of 2 cm^3/min. Assuming that the cone is 12 cm tall and has a radius at the top of 4 cm, find the rate at which the level of the melted ice cream is dropping when the level is 3 cm.

HINT: The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.