10. Properties of Limits

10.1. Limit laws

The following formulas express limits of functions either completely or in terms of limits of their component parts. The formulas are verified by using the precise definition of the limit. (See 9.2 for the verifications of the first two formulas; the verifications of the remaining formulas are omitted.)

LIMIT LAWS. For any functions f and g, and any real numbers c and r, we have (usually writing lim for $\lim\limits_{x\to a}$)

(i) $\lim c = c$,

(ii) $\lim\limits_{x\to a} x = a$,

(iii) $\lim(cf(x)) = c \lim f(x)$,

(iv) $\lim(f(x) + g(x)) = \lim f(x) + \lim g(x)$,

(v) $\lim(f(x)g(x)) = (\lim f(x))(\lim g(x))$,

(vi) $\lim(f(x)/g(x)) = (\lim f(x))/(\lim g(x))$,

(vii) $\lim(f(x))^r = (\lim f(x))^r$,

where it is assumed that on the right the limits exist and the expressions are defined.

Law (vii), applied with $r = 1/n$, says that $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}$.

10.1.1 Example Assume that $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = -3$. Use limit laws to find the following limits:

(a) $\lim\limits_{x\to a}(f(x) + g(x))$.

(b) $\lim\limits_{x\to a}(2g(x))$.

(c) $\lim\limits_{x\to a} \dfrac{\sqrt{f(x)}}{g(x)}$.

Solution We write lim for $\lim_{x\to a}$.

(a) Using limit law (iv), we have $\lim(f(x) + g(x)) = \lim f(x) + \lim g(x) = 4 + (-3) = 1$.

(b) Using limit law (iii), we have $\lim(2g(x)) = 2 \lim g(x) = 2(-3) = -6$.

(c) Using limit laws (vi) and (vii), we have

$$\lim \frac{\sqrt{f(x)}}{g(x)} = \frac{\lim \sqrt{f(x)}}{\lim g(x)} = \frac{\sqrt{\lim f(x)}}{\lim g(x)} = \frac{\sqrt{4}}{-3} = \frac{2}{-3}.$$

□

10.2. Limit of polynomial

10.2.1 Example Use the limit laws and 9.2 to show that, for any a,

$$\lim_{x\to a} 2x^2 - 5x + 4 = 2a^2 - 5a + 4.$$

Solution We have

$$\begin{aligned}\lim_{x\to a} 2x^2 - 5x + 4 &= \lim_{x\to a} 2x^2 + \lim_{x\to a}(-5)x + \lim_{x\to a} 4 && \text{(by (iv))}\\ &= 2\lim_{x\to a} x^2 + (-5)\lim_{x\to a} x + 4 && \text{(by (iii) and (i))}\\ &= 2(\lim_{x\to a} x)^2 + (-5)\lim_{x\to a} x + 4 && \text{(by (vii))}\\ &= 2a^2 - 5a + 4 && \text{(by (ii)).}\end{aligned}$$

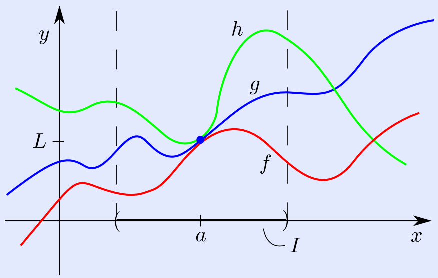
□

The example shows that, for the particular polynomial, the limit can be computed by using the substitution rule. If we accept that the steps in the solution can be generalized to handle an arbitrary polynomial, then we can consider the substitution rule verified for all polynomials. (See 6.1 for the substitution rule.)

In turn, limit law (vi) implies that the substitution rule is valid for all rational functions (i.e., quotients of polynomials). More generally, the limit laws can be used to show that the substitution rule is valid for an expression built up of simpler expressions if it is known that the substitution rule is valid for those simpler expressions.

10.3. Squeeze theorem

Let f, g, and h be functions with graphs as indicated below. The graph of g is trapped between those of f and h, or at least this is the case when x is restricted to the interval I. As x approaches a, the heights of the outside graphs approach the same height L and this forces the middle graph to approach the height L as well.



We state this more formally.

> Squeeze theorem. Let f, g, and h be functions and assume that
>
> $$f(x) \leq g(x) \leq h(x)$$
>
> for all x in some interval I containing the number a (except possibly at $x = a$ where none of the functions need even be defined). If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

10.3.1 Example Use the squeeze theorem to show that

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Solution Using the fact that $-1 \leq \sin(1/x) \leq 1$, we have

$$-x^2 \leq x^2 \sin(1/x) \leq x^2$$

for all x in the interval $I = (-1, 1)$ with 0 removed (this is just one of many choices for I). Now $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$ by the substitution rule (see 6.1), so $\lim_{x\to 0} x^2 \sin(1/x) = 0$ by the squeeze theorem. □

10 – Exercises

10 – 1 Assume that $\lim_{x\to a} f(x) = -2$ and $\lim_{x\to a} g(x) = 6$. Use limit laws to find the following limits:

(a) $\displaystyle\lim_{x\to a} \frac{g(x)}{f(x)}$.

(b) $\displaystyle\lim_{x\to a} (f(x))^{-3}$.

(c) $\displaystyle\lim_{x\to a} \sqrt[3]{f(x) - g(x)}$.

10 – 2 Use the squeeze theorem (10.3) to show that

$$\lim_{x\to 0} (1 + |x|\cos(2\pi/x)) = 1.$$