20. Product rule, Quotient rule

20.1. Product rule

We have seen that the derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[(g(x)\right].$$

One might expect from this that the derivative of a product is the product of the derivatives. This is *not* the case, however. In fact, it usually happens that

$$\frac{d}{dx}\left[f(x)g(x)\right] \neq \frac{d}{dx}\left[f(x)\right]\frac{d}{dx}\left[g(x)\right].$$

For instance,

$$\frac{d}{dx}\left[xx\right] = \frac{d}{dx}\left[x^2\right] = 2x \neq 1 = (1)(1) = \frac{d}{dx}\left[x\right]\frac{d}{dx}\left[x\right].$$

Instead, the rule for finding the derivative of a product is as follows:

PRODUCT RULE. For functions f and g,

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x) + f(x)\frac{d}{dx}\left[g(x)\right].$$

In words, the derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.

For example,

$$\begin{aligned}\frac{d}{dx}\left[x^3\sin x\right] &= \frac{d}{dx}\left[x^3\right]\sin x + x^3\frac{d}{dx}\left[\sin x\right]\\ &= 3x^2\sin x + x^3\cos x.\end{aligned}$$

With $p(x) = f(x)g(x)$, the rule says that $p'(x) = f'(x)g(x) + f(x)g'(x)$, so we verify the rule by establishing this equation using the definition of the derivative:

$$\begin{aligned}p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}g(x+h)+f(x)\frac{g(x+h)-g(x)}{h}\right)\\ &= \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\lim_{h\to 0}g(x+h)+f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\\ &= f'(x)g(x)+f(x)g'(x).\end{aligned}$$

20.1.1 Example Find the derivatives of the following functions:

(a) $f(x) = (x^8 + 2x - 3)e^x$.

(b) $f(t) = 5^t\cos t + 4t^2$.

Solution

(a)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(x^8+2x-3)e^x\right] \\ &= \frac{d}{dx}\left[x^8+2x-3\right]e^x + (x^8+2x-3)\frac{d}{dx}\left[e^x\right] \\ &= (8x^7+2)e^x + (x^8+2x-3)e^x \\ &= (x^8+8x^7+2x-1)e^x. \end{aligned}$$

(b) Here, we need to use the sum rule before using the product rule:

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left[5^t\cos t + 4t^2\right] \\ &= \frac{d}{dt}\left[5^t\cos t\right] + \frac{d}{dt}\left[4t^2\right] \\ &= \frac{d}{dt}\left[5^t\right]\cos t + 5^t\frac{d}{dt}\left[\cos t\right] + 8t \\ &= (5^t\ln 5)\cos t + 5^t(-\sin t) + 8t \\ &= 5^t(\ln 5)\cos t - 5^t\sin t + 8t. \end{aligned}$$

□

The product rule extends naturally to handle any number of factors. For instance,

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x)h(x) + f(x)\frac{d}{dx}\left[g(x)\right]h(x) + f(x)g(x)\frac{d}{dx}\left[h(x)\right].$$

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The derivative is obtained by taking the derivative of one factor at a time, leaving the other factors unchanged, and then summing the results. This rule is verified by using the product rule repeatedly (see Exercise 20 – 3).

20.1.2 Example Find the derivative of $f(x) = (x^3 - 4x^2)e^x \cos x$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(x^3 - 4x^2)e^x \cos x\right] \\ &= \frac{d}{dx}\left[x^3 - 4x^2\right] e^x \cos x + (x^3 - 4x^2)\frac{d}{dx}\left[e^x\right] \cos x \\ &\qquad + (x^3 - 4x^2)e^x \frac{d}{dx}\left[\cos x\right] \\ &= (3x^2 - 8x)e^x \cos x + (x^3 - 4x^2)e^x \cos x + (x^3 - 4x^2)e^x(-\sin x) \\ &= (x^3 - x^2 - 8x)e^x \cos x - (x^3 - 4x^2)e^x \sin x. \end{aligned}$$

□

20.2. Quotient rule

Next, we get the rule for finding the derivative of a quotient.

Quotient rule. For functions f and g,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}\left[f(x)\right] - f(x)\dfrac{d}{dx}\left[g(x)\right]}{(g(x))^2}.$$

In words, the derivative of a quotient is the bottom times the derivative of the top minus the top times the derivative of the bottom, over the bottom squared.

The verification (omitted) is very similar to that for the product rule.

20.2.1 Example Find the derivatives of the following functions:

(a) $f(x) = \dfrac{x^4 - 2x^3 + 8}{x^7 - x}$.

(b) $f(t) = \dfrac{3\sin t}{t^2 - e^t}$.

Solution

(a)

$$f'(x) = \frac{d}{dx}\left[\frac{x^4 - 2x^3 + 8}{x^7 - x}\right]$$
$$= \frac{(x^7 - x)\frac{d}{dx}\left[x^4 - 2x^3 + 8\right] - (x^4 - 2x^3 + 8)\frac{d}{dx}\left[x^7 - x\right]}{(x^7 - x)^2}$$
$$= \frac{(x^7 - x)(4x^3 - 6x^2) - (x^4 - 2x^3 + 8)(7x^6 - 1)}{(x^7 - x)^2}.$$

(b)

$$f'(t) = \frac{d}{dt}\left[\frac{3\sin t}{t^2 - e^t}\right]$$
$$= \frac{(t^2 - e^t)\frac{d}{dt}\left[3\sin t\right] - (3\sin t)\frac{d}{dt}\left[t^2 - e^t\right]}{(t^2 - e^t)^2}$$
$$= \frac{(t^2 - e^t)(3\cos t) - (3\sin t)(2t - e^t)}{(t^2 - e^t)^2}.$$

□

Sometimes a quotient to be differentiated can be rewritten in such a way that the quotient rule becomes unnecessary. In this case, going ahead and rewriting is usually preferable to using the quotient rule; the quotient rule should be used only if it cannot be avoided.

20.2.2 Example Find the derivative of the function

$$f(x) = \frac{x^3 - 5x + 4\sqrt{x}}{x}.$$

Solution At the appropriate step, the function is rewritten in order to avoid using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^3 - 5x + 4\sqrt{x}}{x}\right] \\ &= \frac{d}{dx}\left[x^2 - 5 + 4x^{-1/2}\right] \\ &= 2x - 2x^{-3/2} \\ &= 2x - \frac{2}{(\sqrt{x})^3}. \end{aligned}$$

□

20 – Exercises

20 – 1 Let $f(x) = (x^2 - 3x + 1)(x^4 + 9x^2)$.

(a) Find the derivative of f by first expanding the right-hand side so as to avoid using the product rule.

(b) Find the derivative of f by using the product rule and verify that the result is the same as that obtained in part (a).

20 – 2 Find the derivatives of the following functions:

(a) $f(x) = (2x^5 - 3x^2 + 1)\sin x$

(b) $f(t) = 3\sqrt{t}e^t - \dfrac{5}{t}$

20 – 3 Verify the formula

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x)h(x) + f(x)\frac{d}{dx}\left[g(x)\right]h(x) + f(x)g(x)\frac{d}{dx}\left[h(x)\right].$$

HINT: Apply the product rule viewing $f(x)g(x)h(x)$ as a product with the two factors $f(x)$ and $g(x)h(x)$.

20 – 4 Find the derivative of $f(x) = (2x - 5 + \sqrt[3]{x})8^x \cos x$.

HINT: Use the formula in Exercise 20 – 3.

20 – 5 Let $f(x) = \dfrac{x^4 + 3\sqrt{x} - 5}{x}$.

(a) Find the derivative of f by first rewriting the right-hand side so as to avoid using the quotient rule.

(b) Find the derivative of f by using the quotient rule and verify that the result is the same as that obtained in part (a).

20 – 6 Find the derivatives of the following functions:

(a) $f(x) = \dfrac{3x^2}{5x - 7}$

(b) $f(t) = \dfrac{4\cos t - 1}{2 + 3e^t}$