9 Precise definition of limit

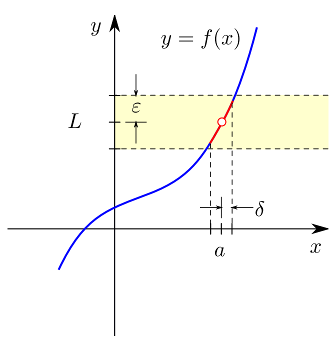
In 5.1, the limit $\lim_{x\to a} f(x)$ was informally defined to be the value (if any) the height of the graph of f approaches as x gets ever closer to a. While this definition conveys an intuitive idea of what is meant by the limit, its use of vague terms such as "approaches" and "closer" keeps it from being rigorous enough to make solid statements about properties of the limit. In this section, we give the precise definition.

9.1 Graph version of definition

The definition is a prescription for deciding when a given number L is to be called the limit $\lim_{x\to a} f(x)$. In terms of the graph of f it goes like this:

Precise definition of limit (graph version). We write $\lim_{x\to a} f(x) = L$ and say that L is the **limit** of the function f as x approaches a if the following condition is satisfied:

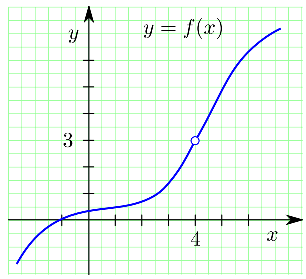
> Given any altitude band centered at L, there exists an interval centered at a such that the part of the graph corresponding to the deleted interval stays within the altitude band. (By "deleted interval" we mean the interval with a removed.)



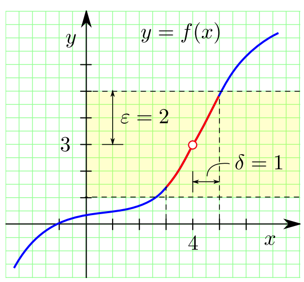
As indicated in the diagram, an altitude band centered at L is determined by specifying a positive number ε (epsilon) and similarly, an interval centered at a is determined by specifying a positive number δ (delta). So, the condition can be expressed succinctly by saying: Given any positive number ε, there exists a positive number δ that "works."

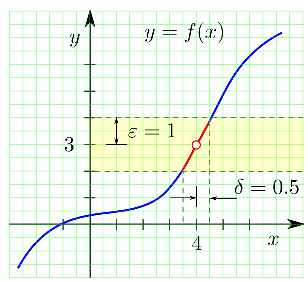
9.1.1 Example For the function f with graph as pictured and the values

$a = 4$ and $L = 3$, find a δ that works for the choice $\varepsilon = 2$, and also a δ that works for the choice $\varepsilon = 1$.



Solution If $\varepsilon = 2$, then $\delta = 1$ works (graph on left). If $\varepsilon = 1$, then $\delta = 0.5$ works (graph on right).



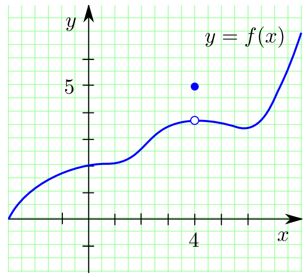


(Incidentally, $\delta = 0.5$ works for both $\varepsilon = 2$ and $\varepsilon = 1$, as does any smaller δ.) □

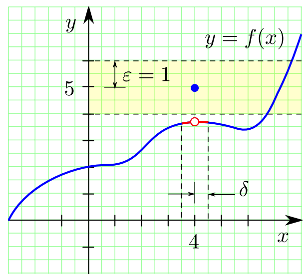
In order for L to be called the limit, it must be the case that for *every* ε, there is a δ that works. In the example, we found δ's that worked for two particular ε's, but it is evident that no matter what ε is chosen to be there will be a corresponding δ that works. Therefore, $\lim_{x\to 4} f(x) = 3$ according to the definition. In this case, anyway, we see that a limit given by the precise definition coincides with our informal idea of the limit as a value that the height of the graph above x approaches as x gets ever closer to 4.

It is sometimes easiest to understand how the precise definition relates to our informal understanding of a limit by looking at a value L that fails to satisfy the definition.

9.1.2 Example For the function f with graph as pictured and the values $a = 4$ and $L = 5$, find an ε for which no δ works.



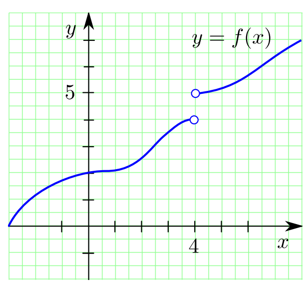
Solution If $\varepsilon = 1$, then, no matter what δ is, the portion of the graph above the deleted interval will not lie inside the altitude band:



□

The example shows that $\lim_{x\to 4} f(x) \neq 5$ according to the definition, and this agrees with our informal idea that the height of the hole should be the limit instead of 5.

9.1.3 Example Let f be the function with graph as pictured and let $a = 4$ and $L = 5$:



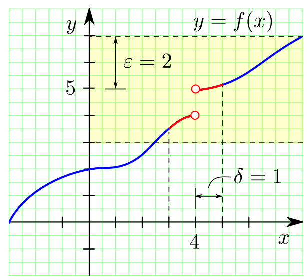
(a) Find a δ that works for the choice $\varepsilon = 2$. Does this imply that $\lim_{x\to 4} f(x) = 5$?

(b) Find an ε for which no δ works. Does this imply that $\lim_{x\to 4} f(x) \neq 5$? Does this imply that $\lim_{x\to 4} f(x)$ does not exist?

(c) Using the precise definition of the limit, argue that $\lim_{x\to 4} f(x)$ does not exist.

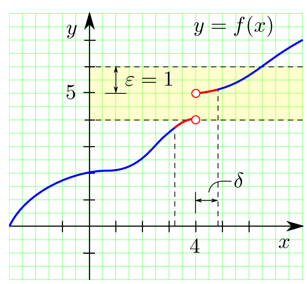
Solution

(a) If $\delta = 1$, then the part of the graph above the corresponding deleted interval stays within the altitude band determined by $\varepsilon = 2$, so this δ works:



This does not imply, however, that $\lim_{x\to 4} f(x) = 5$ since, in order to say this one has to show that for *every* ε, there exists a δ that works.

(b) No matter what δ is, the portion of the graph above the corresponding deleted interval and to the left of 4 will lie outside the altitude band determined by $\varepsilon = 1$, so this is such an ε:



This implies that $\lim_{x\to 4} f(x) \neq 5$ according to the definition. However, it does *not* imply that $\lim_{x\to 4} f(x)$ does not exist, since, for all we know, there might be another choice for L that *does* satisfy the definition.

(c) For any L greater than or equal to 5, there is no δ that works for the choice $\varepsilon = 1$ (same reason as that given in part (b)). For any L less than or equal to 4, there is no δ that works for the choice $\varepsilon = 1$ (again, same reason as that given in part (a), except that the portion of the graph to the right of $x = 4$ will lie outside the altitude band). Finally, for any L between 4 and 5, an altitude band can be drawn that stays within the gap and no δ can work for such an altitude band (this time, the portions of the graph to either side of $x = 4$ will lie outside the band).

We have considered all possibilities for L and have found that none satisfies the definition. Therefore, we conclude that $\lim_{x\to 4} f(x)$ does not exist.

□

9.2 Two limit laws

We can use the precise definition of the limit (graph version) to establish two useful limit laws:

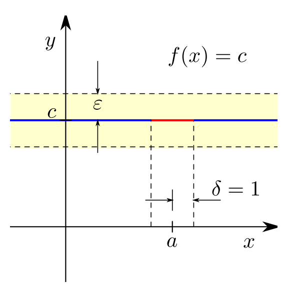
For any real numbers a and c,

(i) $\lim\limits_{x\to a} c = c$,

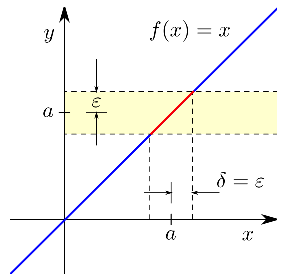
(ii) $\lim\limits_{x\to a} x = a$.

Verifications:

(i) Here, $f(x) = c$ and $L = c$ in the definition of the limit. The graph of f is a horizontal line at height c. Given any ε, the corresponding altitude band includes the entire graph, so $\delta = 1$ will work (or, in fact, so will any δ).



(ii) Here, $f(x) = x$ and $L = a$ in the definition of the limit. The graph of f is the 45° line $y = x$. If ε is given, then $\delta = \varepsilon$ works.



9.3 Analytical version of definition

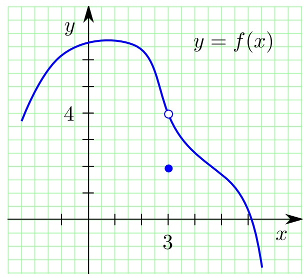
The precise definition of the limit can be stated analytically (i.e., without appealing to a graph):

> Precise definition of limit. We write $\lim_{x\to a} f(x) = L$ and say that L is the **limit** of the function f as x approaches a if the following condition is satisfied:
>
> For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for every x satisfying $0 < |x - a| < \delta$, $f(x)$ satisfies $|f(x) - L| < \varepsilon$.

This is the same as the definition given earlier since saying that x satisfies $0 < |x-a| < \delta$ is the same as saying that x lies in the deleted interval determined by δ and saying $f(x)$ satisfies $|f(x)-L| < \varepsilon$ is the same as saying that the point $(x, f(x))$ on the graph corresponding to x lies in the altitude band determined by ε.

9 – Exercises

9 – 1 For the function f with graph as pictured and the values $a = 3$ and $L = 4$, find a δ that works for the choice $\varepsilon = 2$, and also a δ that works for the choice $\varepsilon = 1$.



9 – 2 For the function f with graph as pictured, use the precise definition of limit (graph version) to show that $\lim_{x \to 5} f(x) \neq 4$.