16. Power rule

Computing a derivative directly from the derivative is usually cumbersome. Fortunately, rules have been discovered for finding derivatives of the most common functions. The rules are easy to apply and they do not involve the evaluation of a limit.

The first rule we establish is the power rule. It gives the derivative of functions that are powers of x. Here are some examples:

$$\begin{aligned} f(x) &= x^3 & &\Longrightarrow & f'(x) &= 3x^2 \\ f(x) &= x^7 & &\Longrightarrow & f'(x) &= 7x^6 \\ f(x) &= x^{10} & &\Longrightarrow & f'(x) &= 10x^9 \end{aligned}$$

In general,

$$f(x) = x^n \qquad \Longrightarrow \qquad f'(x) = nx^{n-1}.$$

(so bring down the power and multiply by x to the one less power).

16.1. Derivation and Statement

In this section, we derive the simple rule just stated for finding the derivative of a function of the form $f(x) = x^n$.

All of the derivative rules are verified by using the definition of the derivative. Although we will not derive all of the rules, we *do* give the derivation of the power rule (in the special case that n is a positive integer) in order to give the reader an idea of what is involved.

The derivation of the power rule involves applying the definition of the derivative (see 13.1) to the function $f(x) = x^n$ to show that $f'(x) = nx^{n-1}$.

We require the rule for expanding the binomial $(x+h)^n$ by using Pascal's triangle. Here are the expansions for the first few choices of n:

$$
\begin{aligned}
(x+h)^1 &= x+h \\
(x+h)^2 &= x^2+2xh+h^2 \\
(x+h)^3 &= x^3+3x^2h+3xh^2+h^3 \\
(x+h)^4 &= x^4+4x^3h+6x^2h^2+4xh^3+h^4
\end{aligned}
$$

The power of x starts with n and then decreases by one in each successive term, while the power of h starts with 0 (viewing x^4 as x^4h^0, for example) and then increases by one in each successive term. The coefficients of the terms are read off from the appropriate row in Pascal's triangle:

$$
\begin{array}{lccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & & & & & \vdots & & & &
\end{array}
$$

A row of the triangle starts and ends with 1 and each of the other numbers is the sum of its two nearest neighbors in the row above.

In general, we have

$$(x+h)^n = x^n + nx^{n-1}h + a_2x^{n-2}h^2 + \cdots + a_{n-1}xh^{n-1} + h^n.$$

The coefficients on the right come from the nth row of Pascal's triangle. For our purposes, only the first two coefficients matter, so we have written the remaining as $a_2, a_3, \ldots$.

We are now ready to derive the formula for the derivative of $f(x) = x^n$:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + a_2x^{n-2}h^2 + \cdots + a_{n-1}xh^{n-1} + h^n) - x^n}{h} \\
&= \lim_{h \to 0} \frac{h(nx^{n-1} + a_2x^{n-2}h + \cdots + a_{n-1}xh^{n-2} + h^{n-1})}{h} \\
&= \lim_{h \to 0} nx^{n-1} + a_2x^{n-2}h + \cdots + a_{n-1}xh^{n-2} + h^{n-1} \\
&= nx^{n-1}.
\end{aligned}$$

Therefore, we have the following rule:

$$\text{If } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}.$$

For instance, if $f(x) = x^5$, then $f'(x) = 5x^4$ (bring down the power and multiply by x to the one less power).

Although we have verified the rule only in the case that n is a positive integer, it holds for any real number n:

> POWER RULE. For any real number n,
>
> $$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

In the statement, we have introduced the **Leibniz** notation for the derivative:

$$\frac{d}{dx} \qquad \text{means} \qquad \text{“the derivative of } \dots \text{.”}$$

So, in words, the formula says “the derivative of x^n is nx^{n-1}.” The Leibniz notation is convenient here since it keeps us from having to assign a name to the function x^n. We will find that it offers other conveniences as well.

The x in the notation represents the independent variable (the input variable). If the independent variable is called something other than x, then the notation changes accordingly. For instance, we would write

$$\frac{d}{dt}\left[t^n\right] = nt^{n-1}.$$

16.2. Using the power rule

The next example illustrates how to use the power rule. Sometimes a function needs to be rewritten using exponential notation before the power rule can be applied.

16.2.1 Example Find each of the following:

(a) $\dfrac{d}{dx}\left[x^{12}\right]$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{x^7}\right]$

(c) $\dfrac{d}{dt}\left[\sqrt[3]{t^8}\right]$

(d) $\dfrac{d}{d\alpha}\left[\alpha^{\pi}\right]$

Solution

(a) $\dfrac{d}{dx}\left[x^{12}\right] = 12x^{11}$.

(b) $\dfrac{d}{dx}\left[\dfrac{1}{x^7}\right] = \dfrac{d}{dx}\left[x^{-7}\right] = -7x^{-8} = -\dfrac{7}{x^8}$.

(c) $\dfrac{d}{dt}\left[\sqrt[3]{t^8}\right] = \dfrac{d}{dt}\left[t^{8/3}\right] = \frac{8}{3}t^{5/3} = \dfrac{8(\sqrt[3]{t})^5}{3}$.

(d) $\dfrac{d}{d\alpha}\left[\alpha^{\pi}\right] = \pi\alpha^{\pi-1}$.

$\square$

16.3. Two special cases of power rule

Two special cases of the power rule warrant special mention:

- $\frac{d}{dx}[x] = 1,$
- $\frac{d}{dx}[1] = 0.$

The first holds since

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[x^1\right] = 1x^0 = 1,$$

and the second holds since

$$\frac{d}{dx}[1] = \frac{d}{dx}\left[x^0\right] = 0x^{-1} = 0.$$

We might have guessed these formulas: The first says that the function $f(x) = x$ (45° line) has general slope function $f'(x) = 1$. The second says that the function $f(x) = 1$ (horizontal line) has general slope function $f'(x) = 0$.

16 – Exercises

16 – 1 Find each of the following derivatives:

(a) $\dfrac{d}{dx}\left[x^{15}\right]$

(b) $\dfrac{d}{dx}\left[x^{-3}\right]$

(c) $\dfrac{d}{dx}\left[x^{2/5}\right]$

16 – 2 Find each of the following derivatives:

(a) $\dfrac{d}{dt}\left[\dfrac{1}{t^5}\right]$

(b) $\dfrac{d}{du}\left[\sqrt[4]{u^3}\right]$

(c) $\dfrac{d}{dz}\left[\dfrac{z^3}{\sqrt{z}}\right]$