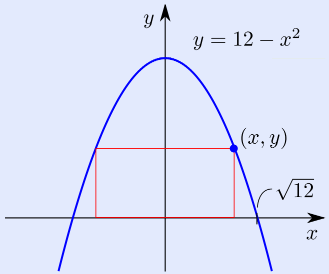
29. Optimization

29.1. Method for solving optimization problems

Here, we use the method of 28 to solve optimization problems.

29.1.1 Example Find the maximum area of a rectangle having base on the x-axis and upper vertices on the parabola $y = 12 - x^2$.

Solution We begin with a diagram:



The quantity we wish to maximize is the area A of the rectangle, which is given by

$$A = (2x)y.$$

Since (x, y) is a point on the parabola, its coordinates satisfy $y = 12 - x^2$. Therefore,

$$A = (2x)\left(12 - x^2\right) = 24x - 2x^3.$$

As can be seen from the diagram, the only values for x that make sense are those between 0 and $\sqrt{12}$, so the problem becomes:

- Find the maximum value of the function $A = 24x - 2x^3$ on the interval $[0, \sqrt{12}]$.

This is a problem to which the method of 28 applies, so we now turn to that method. The derivative of A is $A' = 24 - 6x^2$, which is zero when $x = \pm 2$ and never undefined. Since -2 does not lie in the interval $[0, \sqrt{12}]$, it is omitted. Evaluating, we get

(i) $A|_2 = 24(2) - 2(2)^3 = \boxed{32}$,

(ii) none,

(iii) $A|_0 = 0\left(12 - (0)^2\right) = \boxed{0}$ and $A|_{\sqrt{12}} = 2\sqrt{12}\left(12 - \left(\sqrt{12}\right)^2\right) = \boxed{0}$.

(We chose the most convenient formula for A depending on the input.) The conclusion is that the maximum area of such a rectangle is 32.

(For the interval, we included the "degenerate" cases $x = 0$ (rectangle has width zero) and $x = \sqrt{12}$ (rectangle has height zero). The alternative is to exclude them and work with the open interval $(0, \sqrt{12})$, which requires the computation of limits instead of evaluation at endpoints. For uninteresting degenerate cases such as these it is usually easiest just to include them and work with a closed interval.) □

The main steps in solving an optimization problem, as illustrated above, are as follows:

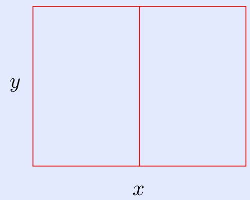
Solving an optimization problem.

- Sketch a diagram and label relevant quantities.
- Write an equation that expresses the quantity to be optimized in terms of the other quantities and use any constraints in the problem to eliminate all but one independent variable.
- Identify an appropriate interval.
- Solve the resulting problem using the method of 28.

29.2. Examples

29.2.1 Example A rectangular pen with a divider down the middle is to be built using 120 m of fencing. Find the dimensions that should be used in order to maximize the area of the enclosed region.

Solution The pen has this configuration:



The quantity to be maximized is the area A of the pen, which is given by the formula $A = xy$. The constraint is that the length of the fence is 120, that is, $2x + 3y = 120$. Solving the constraint for y and substituting into the area formula gives

$$A = x(40 - \tfrac{2}{3}x) = 40x - \tfrac{2}{3}x^2.$$

Since the length of the fencing is 120, we see that x is between 0 and 60. Therefore, the problem can be stated as follows:

- Find the x in the interval $[0, 60]$ that maximizes the function $A = 40x - \frac{2}{3}x^2$.

We use the method of 28. The derivative is $A' = 40 - \frac{4}{3}x$, which is zero when $x = 30$ and is never undefined. Evaluating, we get

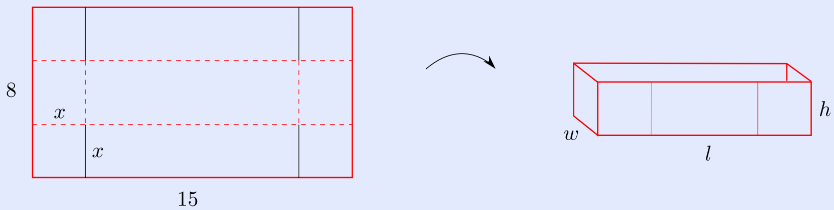
(i) $A|_{30} = 40(30) - \frac{2}{3}(30)^2 = \boxed{600}$,

(ii) none,

(iii) $A|_0 = \boxed{0}$ and $A|_{60} = \boxed{0}$.

Therefore, the maximum (of 600 m^2) occurs when $x = 30$ m and $y = 20$ m (this second dimension coming from the constraint $2x + 3y = 120$). □

29.2.2 Example An origami box is to be made from a 8 cm by 15 cm piece of paper by cutting four slits as shown and folding up the sides. Find the slit length that maximizes the volume of the box.



Solution The quantity to be maximized is the volume V of the box, which is given by $V = lwh$. The constraints are

$$l = 15 - 2x, \qquad w = 8 - 2x, \qquad h = x,$$

so

$$V = (15 - 2x)(8 - 2x)x = (120 - 46x + 4x^2)x = 4x^3 - 46x^2 + 120x.$$

From the picture on the left, we see that x is between 0 and 4 and we include these numbers as degenerate cases. The problem can now be stated as follows:

- Find the x in the interval $[0,4]$ that maximizes the function $V = 4x^3 - 46x^2 + 120x$.

We use the method of 28. The derivative is $V' = 12x^2 - 92x + 120$, which is zero when

$$\begin{aligned} 12x^2 - 92x + 120 &= 0 \\ 3x^2 - 23x + 30 &= 0 \\ (3x-5)(x-6) &= 0, \end{aligned}$$

that is, $x = 5/3$ or $x = 6$. Since 6 is not in the interval $[0,4]$ it is discarded. The derivative is never undefined. Evaluating, we get

(i) $V|_{\frac{5}{3}} = 4\left(\frac{5}{3}\right)^3 - 46\left(\frac{5}{3}\right)^2 + 120\left(\frac{5}{3}\right) = \boxed{\text{some positive number}}$,

(ii) none,

(iii) $V|_0 = \boxed{0}$ and $V|_4 = \boxed{0}$.

Therefore, the maximum volume is obtained by making the slits 5/3 cm long. □

29.2.3 Example Find the positive number that exceeds its natural logarithm by the least amount.

Solution If x is a positive number, then the amount that it exceeds its natural logarithm is $x - \ln x$. Therefore, the problem can be stated as follows:

- Find the x in the interval $(0, \infty)$ that minimizes the function $D = x - \ln x$.

We use the method of 28. The derivative is $D' = 1 - 1/x$, which is zero when $x = 1$ and undefined at $x = 0$. Since 0 is not in the given interval $(0, \infty)$, it is discarded. Evaluating, we get

(i) $D|_1 = 1 - \ln 1 = \boxed{1}$,

(ii) none,

(iii) Since the interval does not include endpoints, we compute appropriate limits:

$$\begin{aligned}\lim_{x\to 0^+} D &= \lim_{x\to 0^+}(x - \ln x) \qquad ((\text{about } 0) - (\text{large neg.}))\\ &= \boxed{\infty}\end{aligned}$$

and

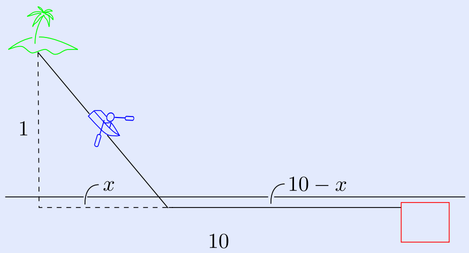
$$\lim_{x\to\infty} D = \lim_{x\to\infty}(x - \ln x) = \boxed{\infty}\ .$$

We have used the graph of the natural logarithm function (see 3.4) to determine these limits. For the first limit, the graph shows that $\ln x$ becomes large and negative as x approaches 0 from the right. For the second limit, since $x - \ln x$ is the difference in height between the 45° line $y = x$ and the graph of $y = \ln x$, we see from the graph of the natural logarithm function that this expression goes to infinity as x goes to infinity as claimed.

Therefore, the minimum (of 1) occurs when $x = 1$. □

29.2.4 Example A small island is located 1 k from a shoreline. Sydney is on the island and she wants to get to a shop that is 10 k down the shore (from the point on the shore closest to the island). She plans to row her boat in a straight line to the shore and then walk the remaining distance to the shop. Find where she should land her boat in order to minimize the travel time assuming that she can row at a rate of 3 k/hr and walk at a rate of 4 k/hr.

Solution The situation looks like this:



The quantity to be minimized is the travel time T. Since distance is rate times time (k=$\left(\frac{\text{k}}{\text{hr}}\right)$(hr)), it follows that time is distance divided by rate. Therefore,

$$\begin{aligned} T &= \text{rowing time} + \text{walking time} \\ &= \frac{\sqrt{x^2+1}}{3} + \frac{10-x}{4} \\ &= \tfrac{1}{3}\left(x^2+1\right)^{1/2} + \tfrac{5}{2} - \tfrac{1}{4}x, \end{aligned}$$

where we have used the Pythagorean theorem to see that the rowing distance is $\sqrt{x^2+1}$. Our intuition tells us that x should be between 0 (row straight toward shoreline) and 10 (row straight toward shop) so that the problem is can be expressed as follows:

- Find the x in the interval $[0, 10]$ that minimizes the function

$$T = \tfrac{1}{3}\left(x^2+1\right)^{1/2} + \tfrac{5}{2} - \tfrac{1}{4}x.$$

We use the method of 28. The derivative is

$$\begin{aligned} T' &= \tfrac{1}{6}\left(x^2+1\right)^{-1/2}(2x) + 0 - \tfrac{1}{4} \\ &= \frac{x}{3\sqrt{x^2+1}} - \frac{1}{4}. \end{aligned}$$

Setting this last expression equal to 0 to find where the derivative is zero, we get

$$\begin{aligned} \frac{x}{3\sqrt{x^2+1}} - \frac{1}{4} &= 0 \\ \frac{x}{3\sqrt{x^2+1}} &= \frac{1}{4} \\ \frac{x^2}{9\left(x^2+1\right)} &= \frac{1}{16} \\ 16x^2 &= 9x^2+9 \\ 7x^2 &= 9 \\ x &= \pm\frac{3}{\sqrt{7}}. \end{aligned}$$

Since $-3/\sqrt{7}$ is not in the interval $[0, 10]$ it is discarded. The derivative is never undefined. Evaluating, we get

(i) $T|_{3/\sqrt{7}} = \tfrac{1}{3}\left((3/\sqrt{7})^2+1\right)^{1/2} + \tfrac{5}{2} - \tfrac{1}{4}(3/\sqrt{7}) = \dfrac{\sqrt{7}+30}{12} \approx \boxed{2.7}$,

(ii) none,

(iii)

$$T|_0 = \frac{\sqrt{(0)^2+1}}{3} + \frac{10-(0)}{4} = \frac{17}{6} \approx \boxed{2.8}$$

(using the earliest formula for T) and

$$T|_{10} = \frac{\sqrt{(10)^2+1}}{3} + \frac{10-(10)}{4} = \tfrac{1}{3}\sqrt{101} \approx \boxed{3.3}\ .$$

Therefore, the minimum travel time (of about 2.7 hours) occurs when $x = 3/\sqrt{7}$ k, so she should land her boat this distance, which is about 1.1 k, down the shoreline. □

29 – Exercises

29 – 1 Find the maximum area of a rectangle having base on the x-axis and upper vertices on the unit circle.

29 – 2 Find the dimensions of a rectangular pen of area 2400 m^2 with a divider down the middle that requires the least length of fencing.

29 – 3 A triangle is to have sides of lengths a and b. Find the angle between these sides that maximizes the area of the triangle.

29 – 4 Find the closest distance that the curve $y = \sqrt{x}$ comes to the point $(2, 0)$.

Hint: The distance D between two points (x_1, y_1) and (x_2, y_2) is given by

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

29–5 A power line is to be run from a power plant on one side of a 1 kilometer wide river to a business on the other side of the river located 4 kilometers downstream. The run is to consist of two parts: the first part is to go straight down the shoreline, costing $50 per meter, while the second part is to go underwater and straight toward the business, costing $100 per meter. Find the distance of the first part of the run that minimizes the total cost.

29–6 Find the dimensions of the lightest open-top cylindrical tin can with volume 1000 cm^3.

HINT: Let ρ denote the weight (in newtons) of 1 cm^2 of tin.