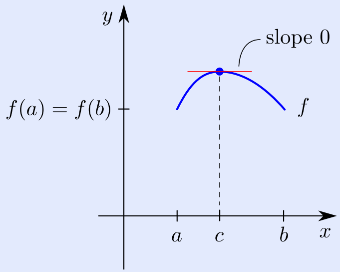
30. Mean value theorem

The mean value theorem is one of the most basic results in calculus. Besides being useful in its own right, it is the key step in proving several other results. We begin with a special case of the mean value theorem known as Rolle's Theorem.

30.1. Rolle's Theorem

ROLLE'S THEOREM. Let f be a continuous function on a closed interval $[a, b]$ such that $f'(x)$ exists for each x between a and b. If $f(a) = f(b)$, then there exists c between a and b such that $f'(c) = 0$.

The verification is as follows: We know that f has a maximum value and a minimum value by the extreme value theorem. Assume that the maximum value occurs at a number c between a and b. Then, since $f'(c)$ is defined (by assumption), the graph must have a horizontal tangent at c. Therefore, $f'(c) = 0$ and the claim of the theorem is met:



A similar argument handles the case where the minimum value occurs between a and b. Finally, if neither the maximum value nor the minimum value occurs between a and b, then they both must occur at the endpoints. But this implies that f must be constant (since $f(a) = f(b)$), so that $f'(x) = 0$ for all x and *any* c between a and b satisfies $f'(c) = 0$.

30.1.1 Example Show that the equation $4x^5 + x^3 + 2x + 1 = 0$ has exactly one (real) solution.

Solution Let $f(x) = 4x^5 + x^3 + 2x + 1$. We are trying to show that there is exactly one a for which $f(a) = 0$.

(At least one such a?) After some reasoned trial and error, we find two input values that yield output values having opposite signs:

$$f(-1) = -6, \qquad f(0) = 1.$$

Since f is continuous, there must be some a between -1 and 0 for which $f(a) = 0$ (see intermediate value theorem in 11).

(At most one such a?) Suppose that there is another number b for which $f(b) = 0$. Then $f(a) = f(b)$, so Rolle's theorem applies to yield a number c between a and b with $f'(c) = 0$.

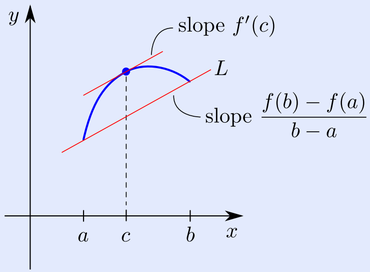
However, $f'(x) = 20x^4 + 3x^2 + 2$, which is always positive, so $f'(c) \neq 0$. We conclude that there can be no other b for which $f(b) = 0$. □

30.2. Mean value theorem

Rolle's theorem requires that $f(a) = f(b)$, that is, the graph of f must have the same height at both endpoints of the interval $[a, b]$. The mean value theorem makes no such assumption:

> MEAN VALUE THEOREM. Let f be a continuous function on a closed interval $[a, b]$ such that $f'(x)$ exists for each x between a and b. There exists c between a and b such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The theorem says that there is a number c between a and b such that the slope of the tangent at c is the same as the slope of the indicated line L:



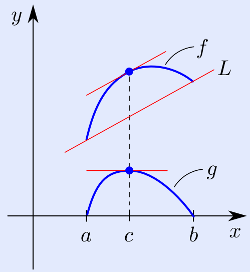
In the special case where $f(a) = f(b)$, the c in the mean value theorem satisfies $f'(c) = 0$ (since the numerator on the right is 0), so Rolle's theorem follows from the mean value theorem. Interestingly, Rolle's theorem is used to prove the mean value theorem. The verification is as follows: The line L has equation

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Solving for y and replacing y by $L(x)$ in order to use function notation for the line, we get

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

If we form a new function g by subtracting L from f, then g satisfies the assumptions of Rolle's theorem and the c given by that theorem is the desired one:



In more detail: Let $g(x) = f(x) - L(x)$. Since $L(a) = f(a)$ and $L(b) = f(b)$, we have $g(a) = 0 = g(b)$. By Rolle's theorem there exists c between a and b such that $g'(c) = 0$. We have

$$\begin{aligned} g'(x) &= f'(x) - L'(x) \\ &= f'(x) - \frac{f(b) - f(a)}{b - a}, \end{aligned}$$

and evaluating at $x = c$ we get

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

so that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

as claimed.

30.2.1 Example Find a number c as in the statement of the mean value theorem, given $f(x) = x^{1/3}$, $a = 0$, $b = 1$.

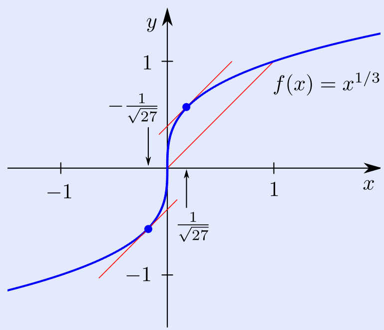
Solution First $f'(x) = \frac{1}{3}x^{-2/3}$. The number c should be between 0 and 1 and should satisfy

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1,$$

that is,

$$\begin{aligned} \tfrac{1}{3}c^{-2/3} &= 1 \\ c^{2/3} &= \tfrac{1}{3} \\ c^2 &= \tfrac{1}{27} \\ c &= \pm\tfrac{1}{\sqrt{27}}. \end{aligned}$$

Since $-1/\sqrt{27}$ is not between 0 and 1, it is of no interest to us. The number $1/\sqrt{27}$, which is a little less than $1/5$, *is* between 0 and 1 and it satisfies the requirements of the theorem, so it is the desired number. Here is the graph:



□

30.2.2 Example At 2:30, Officer A clocks a car traveling at 50 mph in a 55 mph zone. At 3:00, Officer B, located 35 miles down the road, clocks the same car at 50 mph. After communicating by radio, the officers decide to give a speeding ticket. Why?

Solution Let $f(t)$ be the car's position at time t. According to the mean value theorem, at some time c between 2:30 and 3:00,

$$f'(c) = \frac{f(3:00) - f(2:30)}{3:00 - 2:30} = \frac{35 \text{ mi}}{0.5 \text{ hr}} = 70 \text{ mph}.$$

Since $f'(c)$ is the (instantaneous) velocity at time c, which is the speedometer reading at time c, the officers knew that the car was speeding.

(The quotient above is the average velocity of the car between the times 2:30 and 3:00 (see 14), so in this context the mean value theorem says that you cannot average a certain velocity without actually having that velocity at some moment in time.) □

30 – Exercises

30 – 1 Show that the equation $x^3 + 3x = 3$ has exactly one solution.

30 – 2 Find a number c as in the statement of the mean value theorem, given $f(x) = \tan^{-1} x$, $a = -1$, $b = 0$.