24. Logarithmic differentiation

24.1. Statement

The idea of a logarithm arose as a device for simplifying computations. For instance, since $\log ab = \log a + \log b$ (log means $\log_{10}$), one could find the product ab of two numbers a and b by looking up their logarithms in a table, adding those logarithms, and then looking up the antilogarithm of the result using the table again (this last step amounting to raising 10 to the power $\log a + \log b$). Thus, the product (considered difficult) is replaced by a sum (considered easy). Similarly, logarithms replace quotients by differences and powers by products.

Because of computers, logarithms are no longer used to simplify computations with numbers (except within the computer). However, they are still used to simplify expression manipulations as in the method of "logarithmic differentiation" (and they are used in a host of other applications as well).

> Logarithmic differentiation. Given an equation $y = y(x)$ expressing y explicitly as a function of x, the derivative y' is found using **logarithmic differentiation** as follows:
>
> - Apply the natural logarithm ln to both sides of the equation and use laws of logarithms to simplify the right-hand side.
> - Find y' using implicit differentiation.
> - Replace y with $y(x)$.

Statement
Simplifying expressions
Powers with variable base and . . .

The usefulness of the method hinges on the laws of logarithms. We reproduce them here for the special case of base e, which is all that will be required in this section. For positive real numbers x and y and any real number r,

(i) $\ln(xy) = \ln x + \ln y$,

(ii) $\ln(x/y) = \ln x - \ln y$,

(iii) $\ln(x^r) = r \ln x$.

24.2. Simplifying expressions

In the next example, the expression to be differentiated is somewhat complicated. The initial step in the method of logarithmic differentiation simplifies the expression by changing powers to products and products to sums.

24.2.1 Example Given $y = \sqrt{x}e^{x^2}(x^2+1)^{10}$, use the method of logarithmic differentiation to find y' .

Solution Apply ln to both sides and use laws of logarithms:

$$\begin{aligned}
\ln y &= \ln\left(\sqrt{x}e^{x^2}(x^2+1)^{10}\right) \\
&= \ln\left(\sqrt{x}\right) + \ln\left(e^{x^2}\right) + \ln\left((x^2+1)^{10}\right) \\
&= \tfrac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1).
\end{aligned}$$

Find y' using implicit differentiation:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[\tfrac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1)\right]$$
$$\frac{1}{y}\cdot y' = \tfrac{1}{2}\left(\frac{1}{x}\right) + 2x + 10\left(\frac{1}{x^2+1}\right)(2x)$$
$$y' = y\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right).$$

Replace y with $y(x)$:

$$y' = \sqrt{x}e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right).$$

□

The method is somewhat restrictive in that all quantities appearing as the argument of a logarithm must be positive. In the preceding example, the domain of the function is $[0,\infty)$ due to the factor $\sqrt{x}$, so all logarithm arguments are positive for every x in the domain except for $x = 0$. Therefore, the formula obtained for the derivative is valid for all positive x.

24.3. Powers with variable base and variable exponent

We have rules for finding the derivatives of powers with constant exponent, like $(2x+5)^3$, and also for powers with constant base, like $2^{\sin x}$. Logarithmic differentiation provides a means for finding the derivative of powers in which neither exponent nor base is constant.

24.3.1 Example Find the derivative of $y = x^x$ $(x > 0)$.

Solution We use logarithmic differentiation:

$$\begin{aligned} \ln y &= \ln\left(x^x\right) \\ &= x \ln x, \end{aligned}$$

so, using implicit differentiation, we get

$$\begin{aligned} \frac{d}{dx}\left[\ln y\right] &= \frac{d}{dx}\left[x \ln x\right] \\ \frac{1}{y} \cdot y' &= \frac{d}{dx}\left[x\right] \ln x + x \frac{d}{dx}\left[\ln x\right] \\ \frac{1}{y} \cdot y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\ y' &= y(\ln x + 1) \\ y' &= x^x(\ln x + 1). \end{aligned}$$

□

24.3.2 Example Find the derivative of $y = (\ln x)^{\sin x}$ $(1 < x < \pi)$.

Solution We use logarithmic differentiation:

$$\begin{aligned} \ln y &= \ln(\ln x)^{\sin x} \\ &= (\sin x) \ln(\ln x), \end{aligned}$$

so, using implicit differentiation, we get

$$\begin{aligned}
\frac{d}{dx}[\ln y] &= \frac{d}{dx}[(\sin x)\ln(\ln x)] \\
\frac{1}{y}\cdot y' &= \frac{d}{dx}[\sin x]\ln(\ln x) + (\sin x)\frac{d}{dx}[\ln(\ln x)] \\
\frac{1}{y}\cdot y' &= (\cos x)\ln(\ln x) + (\sin x)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) \\
y' &= y\left((\cos x)\ln(\ln x) + \frac{\sin x}{x\ln x}\right) \\
y' &= (\ln x)^{\sin x}\left((\cos x)\ln(\ln x) + \frac{\sin x}{x\ln x}\right).
\end{aligned}$$

□

24 – Exercises

24 – 1 Use logarithmic differentiation to find the derivative of $y = \dfrac{e^{x^3+x}\cos^2 x}{\sqrt{x^4+1}}$ $(-\pi/2 < x < \pi/2)$.

24 – 2 Given $y = (\sin x)^x$ $(0 < x < \pi)$, use logarithmic differentiation to find y'.

24 – 3 Find the derivative of $y = (1+x^2)^{e^x}$.