26. Linear approximation, Leibniz notation

26.1. Linearization

Functions that arise in applications can be quite unwieldy. Given such a function, it is often possible to find a simpler function that behaves enough like the given function that the simpler function can be used instead. Perhaps the simplest function is a linear function, that is, a function having graph a straight line. In this section, we study the process of approximating a function by using a linear function.

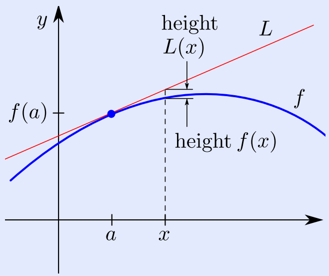
Definition of linearization. Let f be a function, let a be a real number, and assume that $f'(a)$ exists. The **linearization** of f at a is the function

$$L(x) = f(a) + f'(a)(x - a).$$

The linearization L is the line that is tangent to the graph of f at the point $(a, f(a))$ expressed using function notation. Indeed, the tangent line at $(a, f(a))$ passes through this point and has slope $f'(a)$, so its equation is

$$y - f(a) = f'(a)(x - a).$$

Solving for y and replacing y with the function notation $L(x)$ we get the stated formula.



As the diagram shows, if x is close to the base point a, then $L(x)$ (the height to the line above x) is close to $f(x)$ (the height to the curve above x) and therefore provides a good approximation:

$$f(x) \approx L(x) \qquad \text{for } x \text{ close to } a.$$

As x moves away from a however, the approximation can become poor.

26.1.1 Example Use a linearization to approximate $(7.91)^{2/3}$.

Solution With f defined to be $f(x) = x^{2/3}$, the goal is to approximate $f(7.91)$ using a linearization of f. Such a linearization requires the choice of base point a (where the tangent line is drawn). The strategy for picking a is to choose it close to 7.91 so that the approximation will be good, but also choose it to be a number at which f and f' can be easily evaluated. We choose $a = 8$. The corresponding linearization is

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 8^{2/3} + \tfrac{2}{3}(8)^{-1/3}(x-8) \\ &= 4 + \tfrac{1}{3}(x-8). \end{aligned}$$

Therefore,

$$(7.91)^{2/3} = f(7.91) \approx L(7.91) = 4 + \tfrac{1}{3}(7.91 - 8) = 3.97.$$

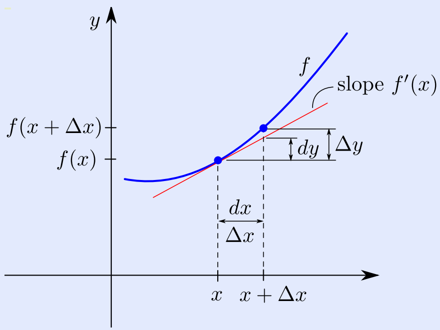
(The actual value of $(7.91)^{2/3}$ to four decimal places is 3.9699.) □

26.2. Differentials

Let $y = f(x)$ be a function and let x be a fixed number. We are interested in how the value of the function changes if we change x by a small amount. If Δx represents the change in x, then the corresponding change in the value of the function, denoted Δy, is the new value minus the old value:

$$\Delta y = f(x + \Delta x) - f(x). \tag{1}$$

As is shown in the following diagram, Δy is the change in height along the curve as x changes by the amount Δx.



This change in height is approximated by the change in height along the tangent line drawn at $(x, f(x))$. When referring to the tangent line, we use dx to represent a change in x and denote by dy the corresponding change in y. The quantities dx and dy are called **differentials**. Since the slope of the tangent line is $f'(x)$, we have

$$dy = f'(x)dx. \tag{2}$$

Therefore, if Δx and dx are taken to be the same change in x (i.e., $\Delta x = dx$) and this change is small, then the change in height along the curve is approximately the same as the change in height along the tangent line:

$$\Delta y \approx dy \qquad \text{for small } \Delta x = dx.$$

The similarity between this statement and the statement $f(x) \approx L(x)$ (for x close to a) made above in the discussion of linearizations is not coincidental. Equation (1) defines the variable Δy as a function of the variable Δx. The graph of this function is the same as the graph of f with the point of tangency $(x, f(x))$ shifted to the origin. Similarly, Equation (2) defines the variable dy as a function of the variable dx and this is the linearization of the function Δy at $a = 0$.

26.2.1 Example Let $f(x) = x^2/4$. Find dy and Δy using $x = 2$ and $dx = 1$ $(= \Delta x)$. Sketch the graph of f and label the distances corresponding to the computed quantities.

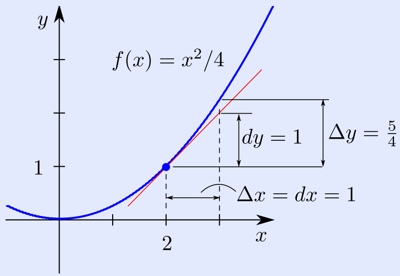
Solution First, $f'(x) = \frac{1}{2}x$. Therefore, when $x = 2$ and $dx = 1$, we have

$$dy = f'(x)dx = f'(2)dx = (1)(1) = 1.$$

Next, when $x = 2$ and $\Delta x = 1$, we have

$$\Delta y = f(x + \Delta x) - f(x) = f(2 + 1) - f(2) = \tfrac{9}{4} - 1 = \tfrac{5}{4}.$$

Here is the sketch:



□

26.3. Leibniz notation

Let $y = f(x)$. When $dx \neq 0$, Equation (2) can be written

$$\frac{dy}{dx} = f'(x).$$

For this reason, $\frac{dy}{dx}$ is often taken as another notation for $f'(x)$ (or, equivalently, y'). This is called the **Leibniz notation** for the derivative of y.

We can now see the evolution of the use of the symbol $\frac{d}{dx}$ to mean "the derivative of." Taking $y = x^2 + 3x - 4$ as an example,

$$\frac{dy}{dx} = \frac{d[x^2 + 3x - 4]}{dx} = \frac{d}{dx}\left[x^2 + 3x - 4\right].$$

Due to the length of the expression $x^2 + 3x - 4$, the last notation is preferred over the middle notation for typographical reasons. However, when the function being differentiated is written using a single letter, that letter is usually moved up next to the d:

$$\frac{d}{dx}[y] \qquad \text{becomes} \qquad \frac{dy}{dx}.$$

Viewing the symbol $\frac{dy}{dx}$ as an actual fraction can serve as a mnemonic device for remembering certain definitions and theorems in calculus:

- The definition of the differential dy (see Equation (2)) becomes

$$dy = \frac{dy}{dx}dx,$$

  so one imagines the dx's on the right canceling.

- Using the definition of the derivative we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) = \frac{dy}{dx},$$

  so $\frac{dy}{dx}$ is what the ratio $\frac{\Delta y}{\Delta x}$ approaches as the change in x becomes ever smaller.

- For functions f and g, put $y = f(g(x))$ and $u = g(x)$. Then $y = f(u)$, and the chain rule says

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = f'(u)g'(x) = \frac{dy}{du}\frac{du}{dx}.$$

Therefore, the chain rule can be written

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

and one imagines the du's on the right canceling.

- For a function f with inverse function f^{-1}, put $y = f(x)$. Then $x = f^{-1}(y)$, and the rule for the derivative of an inverse function says

$$\frac{dx}{dy} = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)} = \frac{1}{\frac{dy}{dx}}.$$

Therefore, the rule for the inverse of an inverse function can be written

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

and one imagines $\frac{dx}{dy}$ as the reciprocal of the fraction $\frac{dy}{dx}$.

26.3.1 Example Use the Leibniz formulation of the chain rule to find the derivative of $y = (2x+5)^3$.

Solution Putting $u = 2x + 5$, we have $y = u^3$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2)(2) = 3(2x+5)^2(2).$$

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