6. Limits by substitution

In 5.2, we computed some limits by looking at graphs. Here, we turn to the problem of computing limits by just looking at expressions defining functions.

6.1. Substitution rule

When it is applicable, the substitution rule provides the easiest way to determine a limit. We introduce the rule by way of an example.

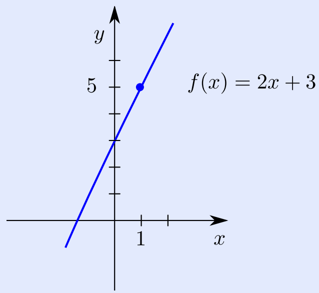
6.1.1 Example Find $\lim_{x \to 1} 2x + 3$.

Solution We are being asked to find what $2x+3$ gets close to as x gets close to 1. Looking at the table

$2x+3$	4.8	4.98	4.998		5.002	5.02	5.2
x	0.9	0.99	0.999	1	1.001	1.01	1.1

we surmise that $\lim_{x\to 1} 2x + 3 = 5$. □

The answer we get agrees with what we find by looking at the associated graph:



What no doubt makes an impression on the reader is that the limit $\lim_{x\to 1} 2x+3$ is precisely what one gets by just substituting 1 for x in the expression $2x + 3$:

$$\lim_{x\to 1} 2x + 3 = 2(1) + 3 = 5.$$

Often (but not always) one can compute a limit by just substituting in this manner:

> Substitution rule. If $f(x)$ is an expression built up from polynomials, roots, absolute values, exponentials, logarithms, trigonometric functions, and/or inverse trigonometric functions by using composition of functions and the operations $+$, $-$, $\times$, $\div$, then for any a for which $f(a)$ is defined, we have $\lim_{x\to a} f(x) = f(a)$; in other words, the limit is found by substituting a in for x.

More briefly, the rule says that you can evaluate the limit of an expression by just replacing the x with the number being approached provided you are not left with something undefined (like a division by zero). This is valid whenever the expression is as described, which is the case for perhaps every expression the reader has encountered (or will encounter).

The substitution rule does not apply to a piecewise-defined function where more than one expression is involved, but we will see in 6.2 that the rule can be used when the number being approached is not a number at which the definition of the function changes.

The verification of the substitution rule for the special case that $f(x)$ is a polynomial is discussed in 10.

6.1.2 Example Find the following limits:

(a) $\lim_{x\to -1} x^3 + 4x^2 - 5.$

(b) $\lim_{x\to 3} \frac{e^{2x}}{\sqrt{x+1}}.$

(c) $\lim_{x\to 2} \cos(\pi x/2) + \ln(x-1).$

(d) $\lim_{x\to 1} \tan^{-1} x.$

Solution

(a) $\lim_{x\to -1} x^3 + 4x^2 - 5 = (-1)^3 + 4(-1)^2 - 5 = -2.$

(b) $\lim_{x\to 3} \frac{e^{2x}}{\sqrt{x+1}} = \frac{e^6}{2}.$

(c) $\lim_{x\to 2} \cos(\pi x/2) + \ln(x-1) = \cos(\pi) + \ln(1) = (-1) + 0 = -1.$

(d) $\lim_{x\to 1} \tan^{-1} x = \tan^{-1} 1 = \pi/4$.

□

6.2. Limit of piecewise-defined function

If the function is defined piecewise, then the substitution rule does not apply to limits approaching values where the definition changes, but away from those values of x it can be appropriate.

6.2.1 Example Let

$$f(x) = \begin{cases} 3 - x^2, & x < 1, \\ x - 1, & x \geq 1. \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x\to 1} f(x)$.

(b) $\lim_{x\to -2} f(x)$.

Solution

(a) In this limit, x is approaching 1, and this is a value where the definition of the function changes, so the substitution rule is not valid. We inspect each one-sided limit. Noting that $f(x)$ is given by the top expression for x to the left of 1 and by the bottom expression for x to the right of 1, we get

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3 - x^2 = 3 - 1 = 2.$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 1 = 1 - 1 = 0.$$

The one-sided limits are not the same, so we conclude that $\lim_{x\to 1} f(x)$ does not exist.

(b) Here, x is getting close to -2, which is away from the value where the definition of the function changes. For x just to either side of -2 (which is all that matters as far as this limit is concerned), we have $x < 1$, so $f(x)$ is given by the top expression, and, since this expression is a polynomial, we can use substitution:

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} 3 - x^2 = 3 - (-2)^2 = -1.$$

□

6 – Exercises

6 – 1 Find the following limits:

(a) $\lim_{x\to -1} x^5 + 2x^4 - 6$.

(b) $\lim_{x\to -3} \dfrac{\sqrt[4]{78-x}}{2^x}$.

(c) $\lim_{x\to 3} \sec(\pi x/4) + \ln(3e/x)$.

(d) $\lim_{x\to -1/2} \sin^{-1} x$.

6 – 2 Let

$$f(x) = \begin{cases} x + 2, & x \leq -1, \\ 2 - (x+1)^2, & -1 < x < 1, \\ 2x - 4, & x > 1. \end{cases}$$

In each case, find the quantity (or explain why it does not exist).

(a) $f(-1)$

(b) $\lim\limits_{x \to -1} f(x)$

(c) $\lim\limits_{x \to -1^+} f(x)$

(d) $\lim\limits_{x \to 1/2} f(x)$

(e) $\lim\limits_{x \to 1} f(x)$