8. Limits by inspection

8.1. The form $\left(\frac{c}{0}\right) \quad (c \neq 0)$

In the limit
$$\lim_{x\to 0^+} \frac{2+x}{x},$$
the numerator is going to 2 while the denominator is going to 0. We say that this limit has **form** $\left(\frac{2}{0}\right)$. This is not an actual fraction. Rather, it is just a way of indicating what the numerator and denominator are going to separately.

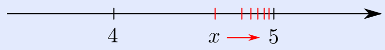
In the limit process, the numerator is getting close to 2 while the denominator is becoming very small. This results in an ever larger fraction. Since both numerator and denominator are always positive (using that x is approaching 0 from the right), we conclude that the limit is ∞. We write
$$\lim_{x\to 0^+} \frac{2+x}{x} \qquad \left(\frac{\text{about } 2}{\text{small pos.}}\right)$$
$$= \infty.$$

(The part in parentheses is what we think; it is not necessary to write it.)

This is called **using inspection** to determine the limit. Inspection can be used whenever the form of the limit is $\left(\frac{c}{0}\right)$ with $c \neq 0$. That is, inspection can be used whenever the denominator is going to 0 and the numerator is *not* going to 0. Inspection cannot be used when the limit has the form $\left(\frac{0}{0}\right)$ (see 8.3 below).

8.1.1 Example Evaluate $\displaystyle\lim_{x\to 5^-} \frac{6}{x-5}$.

Solution As always, we first check to see whether the substitution rule will work. It will not work, due to the zero produced in the denominator. However, the form of the limit is $\left(\frac{6}{0}\right)$, so we can use inspection. Here, x is approaching the number 5 from the left:



In this process, $x-5$ gets ever closer to 0, staying negative the whole time. Therefore, the fraction $6/(x-5)$ gets ever larger in absolute value, staying negative the whole time. We conclude that the limit is $-\infty$. In symbols,

$$\lim_{x\to 5^-} \frac{6}{x-5} \qquad \left(\frac{6}{\text{small neg.}}\right)$$
$$= -\infty.$$

□

The interpretation of the last example in terms of a graph is that the height of the graph of the function $f(x) = 6/(x-5)$ ever decreases as x approaches 5 from the left:

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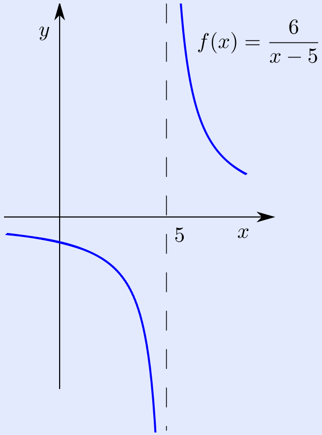

8.1.2 Example Evaluate $\displaystyle\lim_{x\to -2^+} \frac{x-1}{x^3(x+2)}$.

Solution Substitution does not work, but the form of the limit is $\left(\frac{-3}{0}\right)$, so we can use inspection:

$$\lim_{x\to -2^+} \frac{x-1}{x^3(x+2)} \qquad \left(\frac{\text{about } -3}{(\text{about } -8)(\text{small pos.})}\right)$$
$$= \infty.$$

Although the factor x^3 in the denominator gets ever closer to -8, the factor $x+2$ going to zero forces the whole denominator to go to zero. This makes the fraction become ever larger (in absolute value). Because the two negative signs cancel, the fraction is positive and hence approaches ∞. □

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8.2. The form $\left(\frac{c}{\pm\infty}\right)$ $\quad (c \neq \pm\infty)$

The limit

$$\lim_{x\to 0^+} \frac{2+x}{1/x}$$

has the form $\left(\frac{2}{\infty}\right)$. In the limit process, the numerator is getting close to 2 while the denominator is getting very large. This results in a fraction that gets ever closer to 0. Therefore, the limit is 0. We write

$$\begin{aligned} &\lim_{x\to 0^+} \frac{2+x}{1/x} \qquad \left(\frac{\text{about } 2}{\text{large pos.}}\right) \\ &= 0. \end{aligned}$$

Inspection like this can be used whenever a limit has the form $\left(\frac{c}{\pm\infty}\right)$ with $c \neq \pm\infty$. That is, inspection can be used whenever the denominator goes to either ∞ or $-\infty$ and the numerator goes to a particular number (*not* $\pm\infty$). One cannot use inspection when the limit has the form $\left(\frac{\pm\infty}{\pm\infty}\right)$ (see 8.3 below).

(Incidentally, one could also evaluate the above limit by noting that it is the same as $\lim_{x\to 0^+}\left(2x + x^2\right)$ and using the substitution rule.)

8.2.1 Example Evaluate $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \frac{2x}{\tan x}$.

Solution Substitution fails since

$$\tan \pi/2 = \frac{\sin \pi/2}{\cos \pi/2} = \frac{1}{0},$$

which is undefined. However,

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} \qquad \left(\frac{\text{about } 1}{\text{small pos.}}\right)$$
$$= \infty,$$

so the original limit has the form $\left(\frac{\pi}{\infty}\right)$ and we can use inspection:

$$\lim_{x\to\frac{\pi}{2}^-} \frac{2x}{\tan x} \qquad \left(\frac{\text{about } \pi}{\text{large pos.}}\right)$$
$$= 0.$$

□

8.3. When not to use inspection: The forms $\left(\frac{0}{0}\right)$ and $\left(\frac{\pm\infty}{\pm\infty}\right)$

When a limit has the form $\left(\frac{0}{0}\right)$, one cannot use inspection alone to find the limit, as is demonstrated in the following example.

8.3.1 Example Evaluate the following limits:

$$\lim_{x\to 0^+} \frac{x^2}{x}, \qquad \lim_{x\to 0^+} \frac{x}{x^2}, \qquad \lim_{x\to 0^+} \frac{x}{x}.$$

Solution The form of each of these limits is $\left(\frac{0}{0}\right)$. Using inspection alone one might be led to think that, whatever the limits are, they should at least be the same. This is not the case, however, since canceling we get

$$\lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} \frac{x}{1} = 0,$$
$$\lim_{x\to 0^+} \frac{x}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty,$$
$$\lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1.$$

In each case, the numerator getting ever closer to zero is trying to make the fraction small, while the denominator getting ever closer to zero is trying to make the fraction large. So, in a sense, there is a struggle going on. In the first case, the numerator wins and the limit is 0. In the second case, the denominator wins and the limit is ∞. Finally, in the third case, there is a tie and the limit is 1. □

Similarly, when a limit has the form $\left(\frac{\pm\infty}{\pm\infty}\right)$, one cannot use inspection alone to find the limit. In this case, the numerator getting large is trying to make the fraction large, while the denominator getting large is trying to make the fraction small, so again there is a struggle.

The forms $\left(\frac{0}{0}\right)$ and $\left(\frac{\pm\infty}{\pm\infty}\right)$ are examples of "indeterminate forms." We can sometimes use algebraic simplification to handle these forms (see 7 and 12). Later, we will get l'Hôpital's rule, which gives another method for handling these forms (see 31).

8 – Exercises

8–1 Evaluate $\displaystyle\lim_{x\to 4^+} \frac{7}{4-x}$.

8–2 Evaluate $\displaystyle\lim_{x\to 3^-} \frac{x+3}{(x-5)(3-x)}$.