7. Limits by algebraic simplification

The substitution rule (see 6.1) cannot be used to evaluate $\lim_{x \to a} f(x)$ if a is not in the domain of the function f (for instance, if it produces a zero in the denominator).

In this section, we get three methods for evaluating limits when substitution fails. Each method involves an algebraic simplification.

7.1. Factor and cancel method

When substitution produces a zero in both denominator and numerator of a fraction, it sometimes works to factor and cancel.

7.1.1 Example Evaluate $\displaystyle\lim_{x \to -4} \frac{x^2 + 3x - 4}{x + 4}$.

Solution First, due to the zero produced in the denominator, the substitution rule cannot be used. Since the numerator becomes zero as well, we try to factor and cancel:

$$\lim_{x \to -4} \frac{x^2 + 3x - 4}{x + 4} = \lim_{x \to -4} \frac{(x-1)(x+4)}{x + 4} = \lim_{x \to -4} x - 1 = -5.$$

The reason that the next to the last equality is valid is that the fraction equals $x - 1$ except when $x = -4$, and so the limits of these two functions are the same (since the limit is not concerned with the behavior of the function right at -4). □

7.2. Combining fractions method

When the expression you are trying to find the limit of contains a sum or difference of fractions, it sometimes works to combine the fractions. This requires a common denominator, which can always be obtained by just multiplying the two denominators together, though it usually simplifies computations if a *least* common denominator is used.

7.2.1 Example Evaluate $\lim_{x\to 2}\left(\frac{1}{4x-8}-\frac{1}{x^2-4}\right)$.

Solution First, due to the zero produced in either one of the denominators, the substitution rule cannot be used. We combine the fractions by getting a common denominator and then cancel factors to get a function for which the substitution rule is applicable:

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{4x-8}-\frac{1}{x^2-4}\right) &= \lim_{x\to 2}\left(\frac{1}{4(x-2)}-\frac{1}{(x+2)(x-2)}\right)\\ &= \lim_{x\to 2}\left(\frac{x+2}{4(x+2)(x-2)}-\frac{4}{4(x+2)(x-2)}\right)\\ &= \lim_{x\to 2}\frac{(x+2)-4}{4(x+2)(x-2)} = \lim_{x\to 2}\frac{x-2}{4(x+2)(x-2)}\\ &= \lim_{x\to 2}\frac{1}{4(x+2)} = \frac{1}{16}.\end{aligned}$$

□

7.3. Rationalization method

When the expression you are trying to find the limit of is a fraction involving a square root, it sometimes works to rationalize.

The reader probably already knows how to rationalize a numerical denominator:

$$\begin{aligned}\frac{2}{\sqrt{5}+3} &= \frac{2}{\sqrt{5}+3}\cdot\frac{\sqrt{5}-3}{\sqrt{5}-3}\\ &= \frac{2\sqrt{5}-6}{\left(\sqrt{5}\right)^2-3^2}\\ &= \frac{2\sqrt{5}-6}{5-9} = \frac{3-\sqrt{5}}{2}.\end{aligned}$$

The first step is to multiply numerator and denominator by the expression you are trying to rationalize except with the opposite sign (the equality sign is valid since this is merely multiplication by 1). The remaining steps involve simplification.

What makes this process work is the identity $(a+b)(a-b) = a^2 - b^2$. When the left-hand side of this equation is expanded, the middle terms cancel, yielding the right-hand side. This identity (written in the reverse order) is the factorization of a difference of two squares.

In the case of limits, the fraction will involve a variable, but the process is the same. Also, rationalizing the *numerator* is sometimes called for. Finally, one should use the rationalization method only if substitution produces a zero in both denominator and numerator.

7.3.1 Example Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.

Solution First, due to the zero produced in the denominator, the substitution rule cannot be used. Since the numerator becomes zero as well, we use the rationalization method to get a function for which the substitution rule is applicable:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$
$$= \lim_{x\to 0}\frac{(1+x)-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

□

7 – Exercises

7 – 1 Evaluate $\lim_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$.

7 – 2 Evaluate $\lim_{x \to 3} \dfrac{x^2 + x - 12}{x + 6}$.

7 – 3 Evaluate $\lim_{x \to 4^+} \dfrac{\sqrt{x} - 2}{4 - x}$.

7 – 4 Evaluate $\lim_{x \to -\sqrt{2}} \sec^{-1} x$.

7 – 5 Evaluate $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{5}{x^2 + 5x} \right)$.