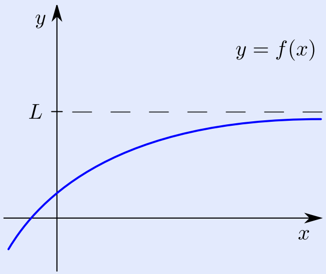
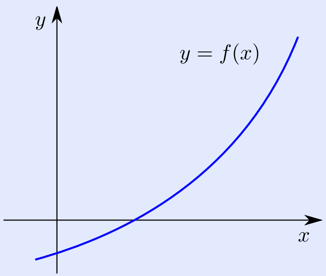
# 12. Limit at infinity

## 12.1. Limit at infinity by inspection

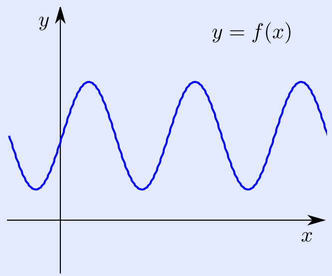
We write $\lim_{x\to\infty} f(x)$ to represent the height that the graph of $f$ approaches (if any) as $x$ gets ever larger. Here are the possibilities:




$$\lim_{x\to\infty} f(x) = L$$




$$\lim_{x\to\infty} f(x) = \infty$$




$$\lim_{x\to\infty} f(x) \text{ does not exist}$$

(If the second graph had turned downward instead, we would have written $\lim_{x\to\infty} f(x) = -\infty$.)

Similarly, $\lim_{x\to-\infty} f(x)$ denotes the height that the graph of $f$ approaches (if any) as $x$ gets ever smaller.

**12.1.1 Example** Use inspection to find the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{5}{x}$

(b) $\displaystyle\lim_{x\to-\infty} 3x^5$

(c) $\displaystyle\lim_{x\to\infty} \cos(2x+3)$

(d) $\displaystyle\lim_{x\to\infty} \cos(1/x)$

(e) $\displaystyle\lim_{x\to-\infty} \tan^{-1} x$

*Solution*

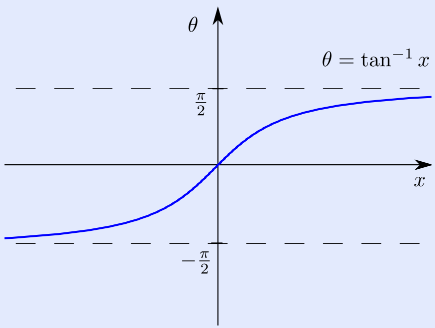
(a) $\displaystyle\lim_{x\to\infty} \frac{5}{x} = 0.$ (We think $\displaystyle\frac{5}{\text{large pos.}}$.)

(b) $\displaystyle\lim_{x\to-\infty} 3x^5 = -\infty.$ (We think $3(\text{large neg.})^5 = \text{large neg.}$.)

(c) $\displaystyle\lim_{x\to\infty} \cos(2x+3)$ does not exist. (The cosine of ever larger numbers oscillates between $-1$ and $1$ and never settles down to any particular value.)

(d) $\displaystyle\lim_{x\to\infty} \cos(1/x) = 1.$ (Since $1/x$ goes to 0, the expression goes to $\cos 0 = 1$.)

(e) $\displaystyle\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$ (The graph of $\tan^{-1} x$ is obtained by reflecting across the 45° line $x = \theta$ the graph of the portion of the graph of $\tan\theta$ corresponding to the reduced domain $(-\pi/2, \pi/2)$):




□

## 12.2. Limit at infinity of polynomial

We cannot use inspection right away to decide the limit

$$\lim_{x\to\infty}\left(-2x^5+x^4+x^2-3\right).$$

The first term of the polynomial is becoming large and negative trying to make the polynomial go to $-\infty$, while the second term is becoming large and positive trying to make the polynomial go to $\infty$. It is like a struggle is going on and it is not clear which term will win (the presence of the other two terms does not make things any better).

To find this limit we factor out the highest power of $x$, namely $x^5$, use a limit law to break the limit up into a product of limits, and then finally use inspection.

**12.2.1 Example** Evaluate $\lim_{x\to\infty}\left(-2x^5+x^4+x^2-3\right)$.

*Solution* The first step is to factor out the highest power of $x$, namely $x^5$:

$$\begin{aligned}\lim_{x\to\infty}\left(-2x^5+x^4+x^2-3\right) &= \lim_{x\to\infty} x^5\left(-2+\frac{1}{x}+\frac{1}{x^3}-\frac{3}{x^5}\right)\\ &= \lim_{x\to\infty} x^5\cdot\lim_{x\to\infty}\left(-2+\frac{1}{x}+\frac{1}{x^3}-\frac{3}{x^5}\right) \qquad ((\text{large pos.})\cdot(\text{about } -2))\\ &= -\infty.\end{aligned}$$

□

**12.2.2 Example** Evaluate $\lim_{x\to-\infty}\left(6-x-4x^3\right)$.

*Solution* We factor out the highest power of $x$, which is $x^3$:

$$\begin{aligned}\lim_{x\to-\infty}\left(6-x-4x^3\right) &= \lim_{x\to-\infty} x^3\left(\frac{6}{x^3}-\frac{1}{x^2}-4\right)\\ &= \lim_{x\to-\infty} x^3\cdot\lim_{x\to-\infty}\left(\frac{6}{x^3}-\frac{1}{x^2}-4\right) \qquad ((\text{large neg.})\cdot(\text{about } -4))\\ &= \infty.\end{aligned}$$

□

Close inspection of these two examples reveals a general principle:

> In finding the limit *at infinity* of a polynomial, all terms other than the one with the highest power of $x$ can be ignored.

For instance,

$$\lim_{x\to\infty}\left(-2x^5+x^4+x^2-3\right)=\lim_{x\to\infty}-2x^5=-\infty,$$

and

$$\lim_{x\to-\infty}\left(6-x-4x^3\right)=\lim_{x\to-\infty}-4x^3=\infty.$$

This technique is not valid unless $x$ is going to either $\infty$ or $-\infty$.

## 12.3. Limit at infinity of quotient

In the limit

$$\lim_{x\to\infty} \frac{1+x+3x^2}{4x^2-5}$$

both numerator and denominator go to infinity, so the limit has the form $\left(\frac{\infty}{\infty}\right)$. It was stated in 8.3 that inspection alone cannot be used to find limits with this form. The numerator getting large is trying to make the fraction large, while the denominator getting large is trying to make the fraction small, so it is like a struggle is going on and it is not clear whether the numerator or the denominator will win (or some compromise will be reached).

To find the limit, we divide both numerator and denominator by the highest power of $x$ that appears in the denominator, namely $x^2$.

**12.3.1 Example** Evaluate $\lim_{x\to\infty} \frac{1+x+3x^2}{4x^2-5}$

*Solution* We divide both numerator and denominator by the highest power of $x$ that appears in the denominator, namely $x^2$:

$$
\begin{aligned}
\lim_{x\to\infty} \frac{1+x+3x^2}{4x^2-5} &= \lim_{x\to\infty} \frac{\dfrac{1+x+3x^2}{x^2}}{\dfrac{4x^2-5}{x^2}} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x^2}+\dfrac{x}{x^2}+\dfrac{3x^2}{x^2}}{\dfrac{4x^2}{x^2}-\dfrac{5}{x^2}} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x^2}+\dfrac{1}{x}+3}{4-\dfrac{5}{x^2}} \\
&= \frac{0+0+3}{4-0} \\
&= \frac{3}{4}.
\end{aligned}
$$

□

To evaluate this limit, we could also use that fact that, for limits at infinity of a polynomial, one can ignore all terms other than the one with the highest power of $x$:

$$
\lim_{x\to\infty} \frac{1+x+3x^2}{4x^2-5} = \lim_{x\to\infty} \frac{3x^2}{4x^2} = \lim_{x\to\infty} \frac{3}{4} = \frac{3}{4}.
$$

**12.3.2 Example** Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^3-x+7}{\sqrt{9x^6+1}}$.

*Solution* As in the previous example, both numerator and denominator are going to infinity, so we cannot use inspection alone. We divide numerator and denominator by the highest power of $x$ appearing in the denominator. This highest power appears to be $x^6$, but, due to the square root sign, the *true* highest power appearing in the denominator is $x^3$, so this is what we divide by:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 - x + 7}{\sqrt{9x^6+1}} &= \lim_{x\to\infty} \frac{\dfrac{x^3 - x + 7}{x^3}}{\dfrac{\sqrt{9x^6+1}}{x^3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{x^3}{x^3} - \dfrac{x}{x^3} + \dfrac{7}{x^3}}{\sqrt{\dfrac{9x^6+1}{x^6}}} \\
&= \lim_{x\to\infty} \frac{1 - \dfrac{1}{x^2} + \dfrac{7}{x^3}}{\sqrt{9 + \dfrac{1}{x^6}}} \\
&= \frac{1 - 0 + 0}{\sqrt{9+0}} \\
&= \frac{1}{3}
\end{aligned}$$

□

To evaluate this limit, we could also use that fact that, for limits at infinity of a polynomial, one can ignore all terms other than the one with the highest power of $x$:

$$\lim_{x\to\infty} \frac{x^3 - x + 7}{\sqrt{9x^6 + 1}} = \lim_{x\to\infty} \frac{x^3}{\sqrt{9x^6}} = \lim_{x\to\infty} \frac{x^3}{3x^3} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}.$$

# 12 – Exercises

12 – 1 Use inspection to find the following limits:

(a) $\lim_{x\to\infty} \frac{-3}{x^2 - 5}$

(b) $\lim_{x\to-\infty} 4 - x^{-3}$

(c) $\lim_{x\to\infty} \frac{\cos x}{x}$

(d) $\lim_{x\to\infty} \tan^{-1}(\pi x - 1)$

(e) $\lim_{x\to-\infty} e^{-x}$

12 – 2 Find the following limit by factoring out the highest power of $x$ that appears:

$$\lim_{x\to\infty} \left(3 - 2x^2 + x^6 - 4x^7\right).$$

12 – 3 Evaluate the following limit by dividing numerator and denominator by the highest power of $x$ that appears in the denominator:

$$\lim_{x\to\infty} \frac{8x^3 - x + 2}{7 - x^2 + 2x^3}.$$

12 – 4 Evaluate the following limit by dividing numerator and denominator by the *true* highest power of $x$ that appears in the denominator:

$$\lim_{x\to\infty} \frac{x^2 - 3}{\sqrt[3]{-8x^6 + 7x^4 - x}}$$

.

12 – 5 Find the following limits by ignoring all terms in each polynomial other than the one with the highest power of $x$. (Note: This works only for limits at infinity.)

(a) $\lim_{x\to-\infty} \left(6x^8 - 7x^3 + 4x - 9\right)$

(b) $\lim_{x\to-\infty} \frac{5x - 6x^3}{2x^3 - 8x^2 + 1}$

(c) $\lim_{x\to\infty} \frac{\sqrt[3]{27x^6 + 5x^4 - 3}}{x^2 - 9}$