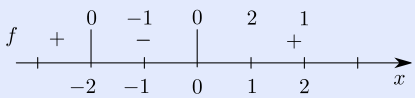
32. Graphing using signs of function and its derivatives

The method of sketching the graph of a function by plotting points is not reliable. It is not possible to plot a point for *every* x value, so there will always be gaps where we are not really sure what is happening.

Here, we sketch the graph of a function by using information about the signs of the function, the signs of its derivative, and the signs of its "second" derivative (meaning the derivative of its derivative).

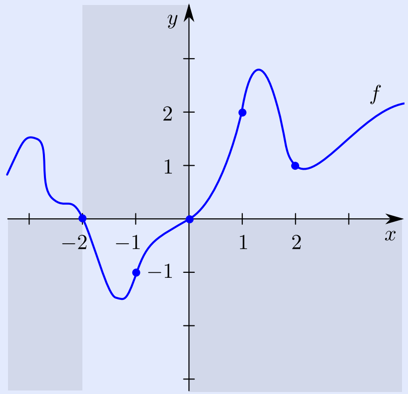
32.1. Signs of function

Let f be a continuous function. For any a, $f(a)$ is the height of the graph of f at a. If $f(a)$ is positive, then the graph is above the x-axis at a; if $f(a)$ is negative, then the graph is below the x-axis at a. Suppose we know that the signs of f are as follows:



The numbers below the line are x values; the signs $+$ and $-$ above the line indicate the signs of the corresponding values of the function, the vertical lines showing the extent to which each sign is valid. The interpretation is that $f(x)$ is positive (graph is above x-axis) for $x < -2$, then $f(x)$ is negative (graph is below x-axis) for $-2 < x < 0$, and finally $f(x)$ is positive (graph is above x-axis) for $x > 0$. Also shown above the number line are actual values of the function for a few select input values. So $f(-2) = 0$, $f(-1) = -1$, $f(0) = 0$, $f(1) = 2$, and $f(2) = 1$.

The graph of f could look something like this:



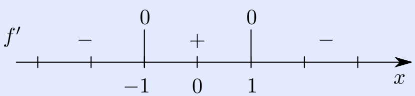
This graph was drawn by first using the signs to block out regions where we know the graph cannot go. Then the points corresponding to the actual given values were plotted. Finally, the rest of the graph was drawn.

This is just one possibility for the graph of f. There are many other graphs that are consistent with the given information, some of which are quite different from the one shown. Although the signs of the function (and the few indicated points) give us a very rough idea of what shape the graph must take, it is desirable to have additional information to use in order to narrow down the possibilities for the graph. Such additional information is provided by the signs of the derivative of f, which we turn to next.

32.2. Signs of derivative

For any a, $f'(a)$ is the slope of the line tangent to the graph of f at a. If $f'(a)$ is positive, then the graph of f rises from left to right at a (we say that f is increasing at a); if $f'(a)$ is negative, then the graph of f falls from left to right at a (we say that f is decreasing at a).

Still working with the function f from the previous section, suppose we know that the signs of f' are as follows:

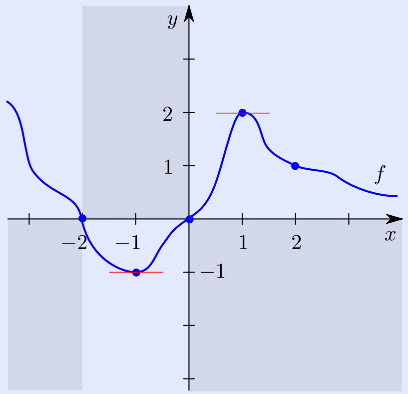


The notations have the same meaning as those for the number line associated with f. For instance, if we wanted to sketch the graph of f', the first $-$ sign would tell us that the graph should be below the x-axis for $x < -1$.

However, we are interested in the graph of the original function f instead, so we interpret the $-$ sign as saying that the graph of f falls from left to right for $x < -1$. Continuing, the $+$ sign says that the graph of f rises from left to right for $-1 < x < 1$, and the last $-$ sign says that the graph of f falls from left to right for $x > 1$. The 0's above the number line say that $f'(-1) = 0$ and $f'(1) = 0$, that is, the graph of f has horizontal tangents at

-1 and 1.

Our first attempt at the graph of f, shown above, violates this information. Amending the graph in order to incorporate the new information, we get the following possibility:

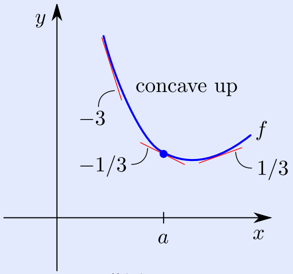


Still, one can imagine other graphs that are consistent with the given information about the signs of the function and its derivative. We get an even further narrowing of the possibilities by considering the signs of the derivative of the derivative.

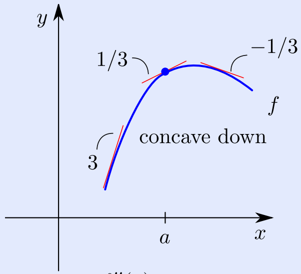
32.3. Signs of second derivative

The derivative of f, namely f', is a function in its own right. Therefore, it makes sense to speak of *its* derivative. Instead of writing $(f')'$ for this derivative, we write f'' and call it the **second derivative** of f. For instance, if $f(x) = 3x^2 + x - 4$, then $f'(x) = 6x + 1$, so that $f''(x) = 6$.

If $f''(a)$ is positive, then f' is increasing at a, which means that the graph of f must look something like that on the left below (we say that the graph is concave up). If $f''(a)$ is negative, then f' is decreasing at a, which means that the graph of f must look something like that on the right below (we say that the graph is concave down).

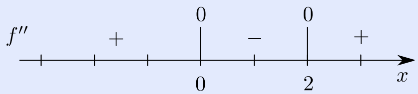


$f''(a)$ is positive

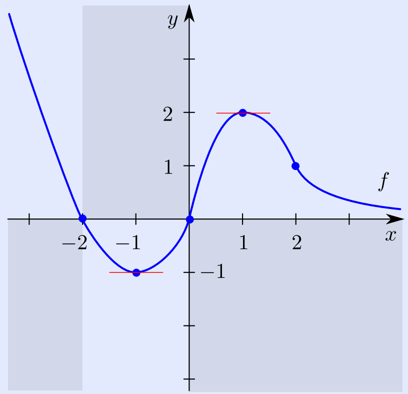


$f''(a)$ is negative

Still working with the function f from the previous sections, suppose we know that the signs of f'' are as follows:



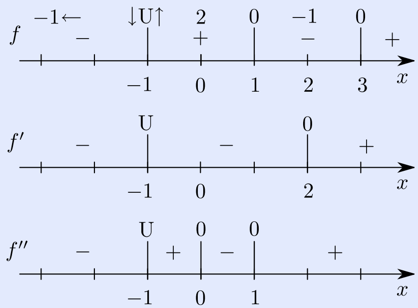
Our most recent attempt at the graph of f violates this information. Making some adjustments in order to comply, we get the following possible graph:



One could still imagine adjustments to the graph that could be made without violating the given information, but such adjustments would have to be slight and we can be confident that our graph is fairly accurate.

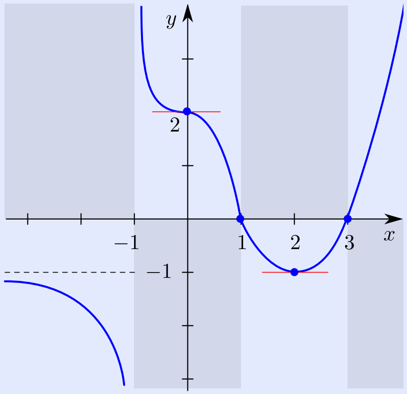
32.4. Examples

32.4.1 Example Sketch the graph of a continuous function f with the properties indicated by the following three number lines:



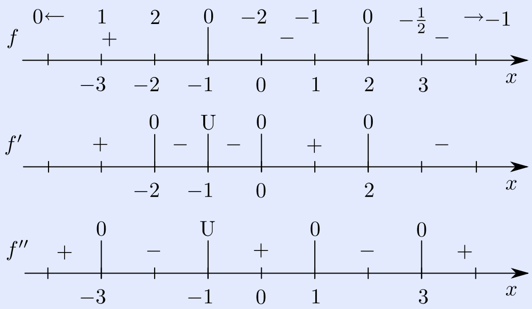
(The letter U indicates a place where the function is undefined. On the f number line it is shown that the graph approaches height -1 as x moves off to the left, the height goes to $-\infty$ as x nears -1 from the left, and the height goes to ∞ as x nears -1 from the right.)

Solution

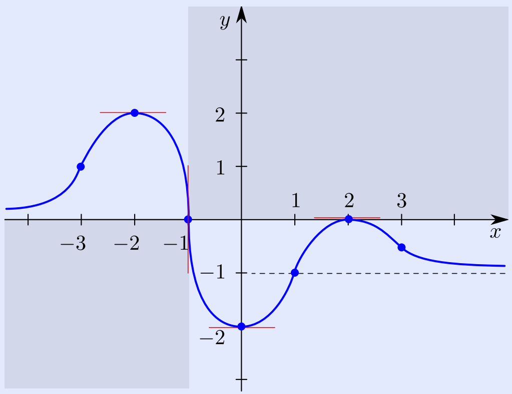


□

32.4.2 Example Sketch the graph of a continuous function f with the properties indicated by the following three number lines:


f
$0\leftarrow$ 1 2 0 -2 -1 0 $-\frac{1}{2}$ $\rightarrow -1$
$+$ $-$ $-$
-3 -2 -1 0 1 2 3 x
f'
0 U 0 0
$+$ $-$ $-$ $+$ $-$
-2 -1 0 2 x
f''
0 U 0 0
$+$ $-$ $+$ $-$ $+$
-3 -1 0 1 3 x

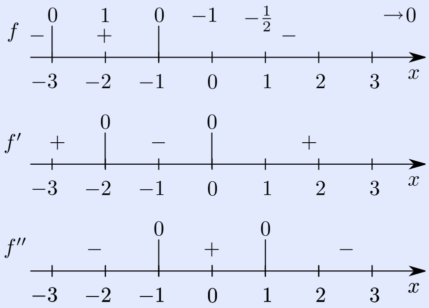
Solution


y
2
1
-1
-2
1 2 3
-3 -2 -1
x

□

32 – Exercises

32 – 1 Sketch the graph of a continuous function f with the properties indicated by the following three number lines:



32 – 2 Sketch the graph of a continuous function f with the properties indicated by the following three number lines: