37. Fundamental theorem of calculus

37.1. Area function is antiderivative

Let \( f(x) = x + 1 \). For any \( x \), let \( F(x) \) denote the area of the region under the graph of \( f \) from 0 to \( x \):

This region is composed of a triangle atop a rectangle, so we can use the familiar area formulas to find that \( F(1) = 3/2 \), \( F(2) = 4 \), and in general

\[
F(x) = \text{area of triangle} + \text{area of rectangle} = \frac{1}{2}x^2 + x.
\]

Of interest to us here is the observation that the derivative of this area function is equal to the original function:

\[
F'(x) = x + 1 = f(x).
\]
Put another way, $F$ is an antiderivative of $f$. This example illustrates a general principle:

**Area function is antiderivative.** Let $f$ be a continuous function and let $a$ be any number. Define the function $F$ by letting

$$F(x) = \text{(signed) area of region between graph of } f \text{ and } x\text{-axis from } a \text{ to } x:$$

Then $F$ is an antiderivative of $f$, that is, $F'(x) = f(x)$.

37.1.1 Example Pictured is the graph of $f(x) = \cos x$. 
(a) By counting squares and portions of squares to estimate areas, make a rough sketch of the function $F$ defined above using $a = 0$ and show that it is consistent with the claim that $F$ is an antiderivative of $f$.

(b) Do the same using $a = -\pi/2$.

**Solution**

(a) The region under the graph of $f$ between 0 and $\pi/2$ occupies two portions of squares and the total area of these two portions appears to be about 1 square unit. Therefore, $F(\pi/2) \approx 1$. Continuing in this fashion, and remembering that areas of regions below the the $x$-axis get counted as negative, we get the following table of values for $F$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$F'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0$</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The graph of $F$ therefore looks like this
which looks like the graph of \( \sin x \). Since \( \sin x \) is an antiderivative of \( \cos x \), this is in agreement with the stated property.

(b) The analysis here is much like that in part \( a \), except that the region that the function \( F \) gives the area of starts at \(-\pi/2\). So, for instance, \( F(0) = 1 \), since the area of the region from \(-\pi/2\) to 0 is 1. We get the table

\[
\begin{array}{c|cccc}
F(x) & 0 & 1 & 2 & 1 & 0 \\
x & \pi/2 & 0 & \pi/2 & \pi & 3\pi/2
\end{array}
\]

The graph of \( F \) therefore looks like this
which looks like the graph of the sine function shifted up one unit. Again, since 
$1 + \sin x$ is an antiderivative of $\cos x$, this is in agreement with the stated property.

Here is a heuristic argument for why the area function $F$ is an antiderivative of $f$: The 
definition of the derivative gives

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}.$$  

Since $F(x + h)$ is the area of the region under the graph of $f$ from $a$ to $x + h$ and $F(x)$ is the 
area of the region from $a$ to $x$, the difference $F(x + h) - F(x)$ is the area of the region from $x$ to $x + h$ (shaded green). This latter is approximately the area of the pictured rectangle, 
which has area $f(x)h$, and the approximation gets better as $h$ gets smaller:
Therefore, we can replace the numerator above by \( f(x)h \) to get

\[
F'(x) = \lim_{h \to 0} \frac{f(x)h}{h} = \lim_{h \to 0} f(x) = f(x),
\]

which says that \( F \) is an antiderivative of \( f \) as claimed.

### 37.2. Fundamental theorem of calculus

We continue to let \( F \) be the area function as in the last section (so \( F(x) \) is the signed area between the graph of \( f \) and the \( x \)-axis from \( a \) to \( x \)). We can express this area function \( F \) using the definite integral, but the way we do so requires some explanation. The definite integral

\[
\int_a^b f(x) \, dx
\]

represents the (signed) area of the region between the graph of \( f \) and the \( x \)-axis from \( a \) to \( b \). The letter \( x \) acts as a dummy variable, meaning that it can be replaced by any other letter without changing the value of the expression (in effect, the area under the graph of \( f \) does not depend on the letter we use for the input variable). Replacing \( x \) by \( t \) frees up the letter \( x \) allowing us to use it as the upper limit of integration:

\[
F(x) = \int_a^x f(t) \, dt.
\]

The property that \( F \) is an antiderivative of \( f \) can now be stated using Leibniz notation:

\[
\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x).
\]

(In short, the derivative is evaluated by replacing the \( t \) in the integrand \( f(t) \) by \( x \).)
37.2.1 Example  Find the following derivatives:

(a) \( \frac{d}{dx} \left[ \int_2^x e^{t^2} \, dt \right] \)

(b) \( \frac{d}{dx} \left[ \int_x^{-3} \sqrt{1 + \sin t} \, dt \right] \)

(c) \( \frac{d}{dx} \left[ \int_1^{\sqrt{x} \tan x} \ln t \, dt \right] \)

Solution

(a) This expression is in the form of the left-hand side of Equation (1), so it equals the integrand with \( t \) replaced by \( x \):

\[
\frac{d}{dx} \left[ \int_2^x e^{t^2} \, dt \right] = e^{x^2}.
\]

(b) As is, the expression is not in the right form to use Equation (1), but we can get the right form by switching the limits of integration and changing the sign (see property of definite integral (ii)):

\[
\frac{d}{dx} \left[ \int_x^{-3} \sqrt{1 + \sin t} \, dt \right] = -\frac{d}{dx} \left[ \int_{-3}^x \sqrt{1 + \sin t} \, dt \right] = -\sqrt{1 + \sin x}.
\]

(c) Since the upper limit of integration is \( \sqrt{x} \tan x \) instead of just \( x \), the expression is not of the right form to use Equation (1) directly. However, the chain rule says
that we can use Equation (1) treating $\sqrt{x}\tan x$ just as though it were $x$ provided we afterwards multiply by the derivative of $\sqrt{x}\tan x$:

$$
\frac{d}{dx} \left[ \int_{1}^{x} \ln t \, dt \right] = \ln (\sqrt{x}\tan x) \cdot \frac{d}{dx} [\sqrt{x}\tan x] \\
= \ln (\sqrt{x}\tan x) \cdot \left( \frac{1}{2}x^{-1/2}\tan x + \sqrt{x}\sec^2 x \right).
$$

\[ \square \]

**Fundamental theorem of calculus.**

(i) \( \frac{d}{dx} \left[ \int_{a}^{x} f(t) \, dt \right] = f(x), \)

(ii) \( \int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} \), where F is any antiderivative of f and

\[ F(x) \bigg|_{a}^{b} = F(b) - F(a). \]

We have already given an argument for why part (i) holds. Here is the verification of part (ii). Let $F$ be any antiderivative of $f$. By part (i), the function $G$ given by

$$
G(x) = \int_{a}^{x} f(t) \, dt
$$
is an antiderivative of $f$ and so $G(x) = F(x) + C$ for some real number $C$ (see 34). Therefore,

$$\int_a^b f(x) \, dx = \int_a^b f(t) \, dt - \int_a^a f(t) \, dt \quad \text{(prop. of integral (i))}$$

$$= G(b) - G(a) \quad \text{(definition of $G$)}$$

$$= (F(b) + C) - (F(a) + C)$$

$$= F(b) - F(a) = F(x) \bigg|_a^b.$$

### 37.2.2 Example

Find $\int_1^5 3x^2 \, dx$.

**Solution** We use part (ii) of the fundamental theorem of calculus with $f(x) = 3x^2$. An antiderivative of $f$ is $F(x) = x^3$, so the theorem says

$$\int_1^5 3x^2 \, dx = x^3 \bigg|_1^5 = 5^3 - 1^3 = 124.$$

We now have an easier way to work Examples 36.2.1 and 36.2.2.

### 37.2.3 Example

(a) Find $\int_0^6 (x^2 + 1) \, dx$. 

(b) Find $\int_a^b x \, dx$.

Solution

(a) We have

$$\int_0^6 (x^2 + 1) \, dx = \left( \frac{x^3}{3} + x \right) \bigg|_0^6 = \left( \frac{6^3}{3} + 6 \right) - \left( \frac{0^3}{3} + 0 \right) = 78,$$

in agreement with Example 36.2.1.

(b) We have

$$\int_a^b x \, dx = \frac{x^2}{2} \bigg|_a^b = \frac{b^2}{2} - \frac{a^2}{2},$$

in agreement with Example 36.2.2.

37.2.4 Example Use a definite integral to show that the unit circle has area $\pi$. (Hint: See Example 34.3.3.)

Solution The unit circle has equation $x^2 + y^2 = 1$. Solving for $y$ and taking the positive square root gives the upper semicircle: $y = \sqrt{1 - x^2}$. The definite integral of this function from 0 to 1 gives the area of the quarter circle in the first quadrant, so, using Example
34.3.3, we get

\[
\text{Area of unit circle } = 4 \int_0^1 \sqrt{1 - x^2} \, dx
\]

\[
= 4 \cdot \frac{1}{2} \left( x\sqrt{1 - x^2} + \sin^{-1} x \right) \bigg|_0^1
\]

\[
= 2(\frac{\pi}{2} - 0) = \pi.
\]

37.2.5 Example Let \( F(x) = \int_1^x (4t - 3) \, dt \).

(a) Find \( F'(x) \) by using part (i) of the fundamental theorem of calculus.

(b) Find \( F'(x) \) by first using part (ii) of the fundamental theorem of calculus to evaluate the integral.

Solution

(a) Part (i) gives

\[
F'(x) = \frac{d}{dx} \left[ \int_1^x (4t - 3) \, dt \right] = 4x - 3.
\]
(b) Part (ii) gives

\[ F(x) = \int_{1}^{x} (4t - 3) \, dt \]

\[ = (2t^2 - 3t) \bigg|_{1}^{x} \]

\[ = (2x^2 - 3x) - (2(1)^2 - 3(1)) \]

\[ = 2x^2 - 3x + 1, \]

so

\[ F'(x) = \frac{d}{dx} \left[ 2x^2 - 3x + 1 \right] = 4x - 3. \]

The notation for the definite integral \( \int_{a}^{b} f(x) \, dx \) is intended to suggest the summing of rectangle areas \( f(x) \, dx \) as \( x \) goes from \( a \) to \( b \); it gives a signed area. We have used the notation \( \int f(x) \, dx \) to denote the most general antiderivative of \( f \), and have called this the indefinite integral of \( f \). Part (ii) of the fundamental theorem of calculus says that

\[ \int_{a}^{b} f(x) \, dx = \int f(x) \, dx \bigg|_{a}^{b}, \]

so the indefinite integral can be thought of as a general area function; evaluating it at two numbers and taking the difference gives the area under the curve between those two numbers.
37.3. Definite integral with substitution

There are two methods for evaluating a definite integral when a $u$-substitution is required:

37.3.1 Example Evaluate $\int_{0}^{1} (5x - 2)^2 \, dx$ two ways:

(a) by changing to $u$-limits;

(b) by changing back to $x$’s before evaluating at the limits.

Solution

(a) The expression $(5x - 2)^2$ is the composition of $5x - 2$ (inside function) and $x^2$ (outside function), so we let $u$ be the inside function:

Let $u = 5x - 2$, so that $du = 5 \, dx$.

At the step where we make the change to $u$’s, we also change the limits of integration to corresponding $u$-limits by using the substitution equation $u = 5x - 2$ (so $x = 0$...
gives \( u = -2 \), and \( x = 1 \) gives \( u = 3 \):

\[
\int_0^1 (5x - 2)^2 \, dx = \frac{1}{5} \int_0^1 (5x - 2)^2 \frac{5 \, dx}{u^2} \\
= \frac{1}{5} \int_{-2}^3 u^2 \, du \quad \text{(change to } u\text{-limits)} \\
= \frac{1}{5} \cdot \frac{u^3}{3} \bigg|_{-2}^3 = \frac{1}{15} \left( u^3 \bigg|_{-2}^3 \right) \\
= \frac{1}{15} \left( 3^3 - (-2)^3 \right) = \frac{1}{15} (35) = \frac{7}{3}.
\]

(b) Here, we use the same \( u\)-substitution as before, but we keep the original limits of integration and just change back to \( x \)'s before using them:

\[
\int_0^1 (5x - 2)^2 \, dx = \frac{1}{5} \int_0^1 (5x - 2)^2 \frac{5 \, dx}{u^2} \\
= \frac{1}{5} \int_{x=0}^{x=1} u^2 \, du \\
= \frac{1}{5} \cdot \frac{u^3}{3} \bigg|_{x=0}^{x=1} = \frac{1}{15} \left( u^3 \bigg|_{x=0}^{x=1} \right) \\
= \frac{1}{15} \left( (5x - 2)^3 \bigg|_0^1 \right) \quad \text{(change back to } x\text{'s)} \\
= \frac{1}{15} \left( 3^3 - (-2)^3 \right) = \frac{1}{15} (35) = \frac{7}{3}.
\]

(If just 0 and 1 are written where we have written \( x = 0 \) and \( x = 1 \), then they are interpreted as being \( u\)-values, which is not correct.)
The first method (changing to \(u\)-limits) is preferred since it requires fewer steps and it allows one to forget about \(x\)'s after the \(u\)-substitution has been made.

37.3.2 Example Evaluate \(\int_{0}^{\sqrt{\pi}} x \cos x^2 \, dx\).

Solution The expression \(\cos x^2\) is the composition of \(x^2\) (inside function) and \(\cos x\) (outside function), so we let \(u\) be the inside function:

Let \(u = x^2\), so that \(du = 2x \, dx\).

We have

\[
\int_{0}^{\sqrt{\pi}} x \cos x^2 \, dx = \frac{1}{2} \int_{0}^{\sqrt{\pi}} \cos x^2 \cdot 2x \, dx \\
= \frac{1}{2} \int_{0}^{\pi} \cos u \, du \quad \text{(change to } u\text{'s, including limits)}
\]

\[
= \frac{1}{2} \sin u \bigg|_{0}^{\pi} = \frac{1}{2} (\sin \pi - \sin 0) = 0.
\]

(The definite integral gives \textit{signed} area, so here the area of the region above the \(x\)-axis is the same as the area of the region below the \(x\)-axis leading to a total signed area of 0.)
37.4. Displacement as definite integral

Suppose an object is moving in a straight line (along the $x$-axis, for instance). Let $f(t)$ be its position at time $t$. The **displacement** of the object from time $t = a$ to time $t = b$ is the final position minus the initial position:

$$\text{displacement} = f(b) - f(a).$$

Let $v(t)$ be the velocity of the object at time $t$.

**Displacement as definite integral.** The displacement from time $t = a$ to time $t = b$ is given by

$$\text{displacement} = \int_a^b v(t) \, dt.$$

Here is the reason: Since $f'(t) = v(t)$ (see 14), $f$ is an antiderivative of $v$. Therefore, part (ii) of the fundamental theorem of calculus says

$$\int_a^b v(t) \, dt = f(b) - f(a) = \text{displacement}.$$

Displacement gives net distance traveled; forward movement gets counted as positive and backward movement gets counted as negative. If it is total distance traveled that is desired, then the time interval should be divided into subintervals on which the velocity is always positive (object is traveling forward) or always negative (object is traveling backward) and the absolute values of the corresponding displacements should be summed.
37.4.1 Example  A car is traveling in a straight line with velocity at time $t$ seconds given by $v(t) = (5 - t)/2$ (in meters per second).

(a) Find the car’s displacement from time $t = 0$ to time $t = 8$.

(b) Find the total distance traveled by the car during the time interval $t = 0$ to $t = 8$.

Solution

(a) Using the formula above, we have

\[
\text{displacement} = \int_0^8 v(t) \, dt = \int_0^8 \left(\frac{5}{2} - \frac{1}{2} t\right) \, dt
\]

\[
= \left[\frac{5}{2} t - \frac{1}{4} t^2\right]_0^8 = (20 - 16) - (0 - 0) = 4,
\]

so the displacement is 4 meters.

(b) The velocity function can change signs only where it is zero or undefined. It is zero when $t = 5$ and is never undefined. On each of the resulting intervals, $[0, 5]$ and $[5, 8]$, the car is either always moving forward or always moving backward, so the total distance traveled is the sum of the absolute values of the displacements on these
The car travels a total of $\frac{17}{2} = 8.5$ meters. (In fact, the last line of our computation shows that the car travels $25/4$ m forward, and then $9/4$ m backward for a net distance traveled of $25/4 - 9/4 = 4$ m, in agreement with part (a).)
If the car had constant velocity 10 m/s and it traveled for 3 s, then its displacement would be simply the product

\[
\text{displacement} = (10 \text{ m/s})(3 \text{ s}) = 30 \text{ m}.
\]

Since the car in the example has variable velocity \( v(t) = (5 - t)/2 \), we cannot just multiply velocity by time to get displacement. However, we can approximate the displacement by taking a reading of the speedometer at the beginning of each, say, two-second interval, and assuming that the velocity remains constantly that value throughout the interval. The table of speedometer readings is

<table>
<thead>
<tr>
<th>( v(t) )</th>
<th>( 5/2 )</th>
<th>( 3/2 )</th>
<th>( 1/2 )</th>
<th>( -1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

So

\[
\text{displacement} \approx \left(\frac{5}{2}\right)(2) + \left(\frac{3}{2}\right)(2) + \left(\frac{1}{2}\right)(2) + \left(-\frac{1}{2}\right)(2) = 8 \text{ m}.
\]

The terms appearing in this formula are the (signed) areas of the four rectangles pictured here:
We have seen that the exact displacement from \( t = 0 \) to \( t = 8 \) is 4 m, so our approximation of 8 m is not very good. This is because we assumed that the velocity stayed constant during each two-second time interval when it in fact changes. A better approximation can be obtained by taking speedometer readings more frequently, say, every second, or, better yet, every one-tenth second. The resulting approximations correspond to a greater number of (narrower) rectangles in the picture. The limit of the approximation as the length \( \Delta t \) of the time interval between readings goes to zero, or, equivalently, as the number \( n = \frac{8}{\Delta t} \) of rectangles goes to infinity, is the exact displacement:

\[
\text{displacement} = \lim_{n \to \infty} A_n = \int_0^8 v(t) \, dt,
\]

where \( A_n \) is the approximation resulting from \( n \) evenly spaced readings. This gives another way of verifying the earlier formula. (The region between the graph of \( v \) and the \( t \)-axis from 0 to 8 is composed of two triangles, so elementary geometry can be used to verify that the signed area of this region is indeed 4 in agreement with our earlier findings for the displacement.)
37 – Exercises

37 – 1 Pictured is the graph of \( f(x) = 2x \).

(a) Let \( F \) be the area function corresponding to \( f \) and \( a \) (see 37). By counting squares and portions of squares to estimate areas, make a rough sketch of \( F \) using \( a = 0 \) and show that it is consistent with the claim that \( F \) is an antiderivative of \( f \).

(b) Do the same using \( a = -2 \).
37 – 2 Find each of the following derivatives:

(a) \[ \frac{d}{dx} \left[ \int_3^x t^{\sqrt{t}} \, dt \right] \]
(b) \[ \frac{d}{dx} \left[ \int_8^{x^2 \ln x} \frac{1}{1 + t^3} \, dt \right] \]
(c) \[ \frac{d}{dx} \left[ \int_x^{2x} \cos t^2 \, dt \right] \]

HINT: For (c), use integral property (iii) with \( c = 0 \) (see 36).

37 – 3 Evaluate \[ \int_1^4 (5x^2 - 3x + 2\sqrt{x}) \, dx \].

37 – 4 Evaluate \[ \int_0^{\sqrt{7}} x^3 \sqrt{x^2 + 1} \, dx \] two ways:

(a) by changing to \( u \)-limits;
(b) by changing back to \( x \)'s before evaluating at the limits.

37 – 5 Evaluate \[ \int_0^\pi 3 \cos^2 x \sin x \, dx \].
A particle travels in a straight line with velocity at time \( t \) seconds given by \( v(t) = 3 \cos t \) (m/s).

(a) Find the displacement of the particle for the time interval \( t = 0 \) to \( t = \frac{3\pi}{4} \).

(b) Find the total distance traveled by the particle over the time interval \( t = 0 \) to \( t = \frac{3\pi}{4} \).