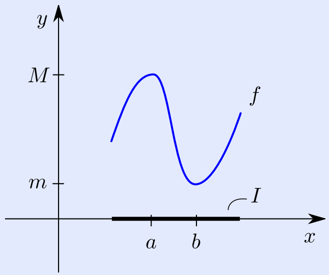
28. Extreme values

28.1. Terminology and main theorem

The function f with graph as pictured below has a maximum value of M and a minimum value of m on the interval I.



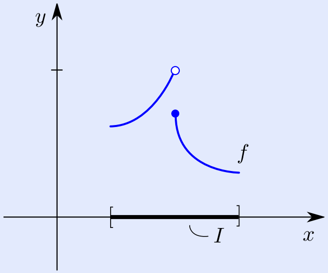
These values are referred to as the **extreme values** of the function f on the interval I. In applications, extreme values of functions are of particular interest (e.g., maximum profit, minimum cost, maximum velocity, minimum carbon emission). The precise definition is as follows:

Maximum value, minimum value. Given a function f and a subset I of the set of real numbers, we say that

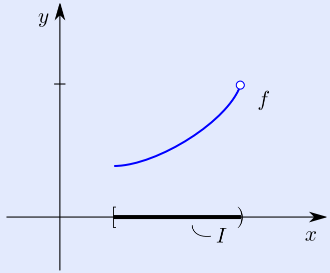
- f has a **maximum value** of M (a real number) if
  (i) $f(a) = M$ for some a in I, and
  (ii) $f(x) \leq M$ for all x in I,
- f has a **minimum value** of m (a real number) if
  (i) $f(b) = m$ for some b in I, and
  (ii) $f(x) \geq m$ for all x in I.

Taking the case of a maximum value M, part (i) says that M is an actual output of the function f, while part (ii) says that no output of f is greater than M. A similar statement holds for the minimum value m.

A function need not have a maximum value on a set I as the following illustrations show:



no maximum



no maximum

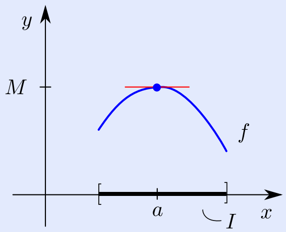
On the left, the height of the hole is not a maximum, since it is not equal to $f(a)$ for any a in the interval I (nor can any other height possibly be a maximum). On the right, the right-hand endpoint is not included in the interval I, so the height of the hole is not a maximum for the same reason as before (and, again, no other height can possibly be a maximum). Similar considerations show that a function need not have a minimum value on a set I.

However, a continuous function on a closed interval always has a maximum value and a minimum value:

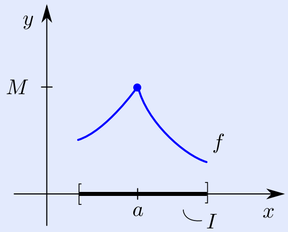
> Extreme value theorem. If f is a continuous function on a closed interval $I = [a, b]$, then f has both a maximum value and a minimum value.

28.2. Finding extremes on a closed interval

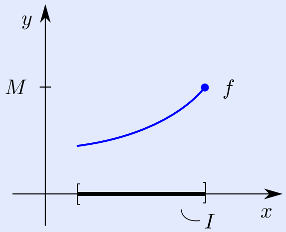
It is one thing to know that extremes exist and another to know how to find them. The method we give for finding the extremes of a continuous function on a closed interval is based on the following illustrations, which show the only possible ways that a maximum value can occur (and similarly for a minimum value):



$f'(a) = 0$



$f'(a)$ undefined



endpoint

FINDING EXTREME VALUES ON A CLOSED INTERVAL. The extreme values of a continuous function f on a closed interval $I = [a, b]$ can be found as follows:

- Evaluate f at

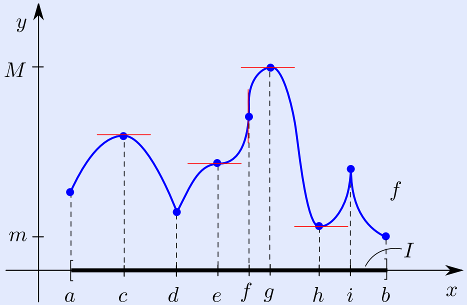
  (i) every x in I for which $f'(x) = 0$,

  (ii) every x in I for which $f'(x)$ is undefined,

  (iii) both endpoints a and b.

- The greatest value obtained above is the maximum value; the least value is the minimum value.

The usefulness of the method comes from the fact that it does not require the graph of the function; the method is carried out using only an expression for the function. However, in order to see why the method works it helps to look at the graph of a representative function. Take, for instance, the function f with the following graph:



The numbers c, e, g, h are where the derivative is zero (since the graph has a horizontal tangent line at each of these numbers) and the numbers d, f, i are where the derivative is undefined (since the graph either has a corner or a vertical tangent line at these numbers). According to the remarks above, these values of x, along with the endpoints a and b, are the only candidates for places at which extreme values can occur. Therefore, if f is evaluated at each of the lettered numbers to get the corresponding heights to the graph, then the greatest value appearing will be the maximum and the least value appearing will be the minimum. Here, the maximum value is M, occurring at $x = g$, and the minimum value is m occurring at the endpoint $x = b$.

Finding where the derivative is zero or undefined will occasionally produce numbers like e and f where there is no chance of an extreme value occurring. Although one can do some further analysis to show that such points can be discarded (like showing that the graph rises both before and after the number, as is the case with both e and f), it is usually easier just to evaluate the function at such points and include these values in the pool of possible maximum values and minimum values.

28.2.1 Example Find the extreme values of the function $f(x) = x^2 + x$ on the interval

$[-2, 2]$ and the x values at which they occur. Also, sketch the graph of f.

Solution The derivative of f is $f'(x) = 2x + 1$, which is zero when $x = -1/2$ and is never undefined. Following the steps outlined above, we have

(i) $f(-1/2) = \boxed{-1/4}$,

(ii) none,

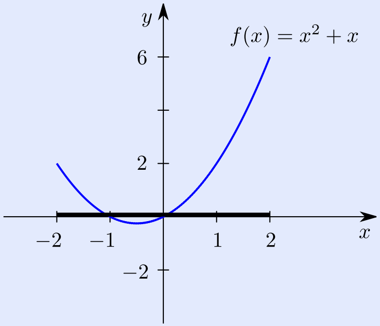
(iii) $f(-2) = \boxed{2}$ and $f(2) = \boxed{6}$.

Therefore,

Maximum: 6, occurring at $x = 2$,

Minimum: -1/4, occurring at $x = -1/2$.

Here is the sketch:



□

28.2.2 Example Find the extreme values of the function $f(x) = \sqrt[3]{x^2 - x}$ on the interval $[0, 2]$ and the x values at which they occur.

Solution We begin by computing the derivative of f:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(x^2 - x)^{1/3}\right] \\ &= \tfrac{1}{3}(x^2 - x)^{-2/3}(2x - 1) \\ &= \frac{2x - 1}{3\left(\sqrt[3]{x^2 - x}\right)^2}. \end{aligned}$$

Therefore, f' is zero when $x = 1/2$ (the only way a fraction can equal zero is if the numerator equals zero), and f' is undefined when the denominator equals zero, that is, when

$$\begin{aligned} x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0 \text{ or } x = 1. \end{aligned}$$

Evaluating f at these x-values and at the endpoints, we get

(i) $f(1/2) = \boxed{-\dfrac{1}{\sqrt[3]{4}}}$,

(ii) $f(0) = \boxed{0}$ and $f(1) = \boxed{0}$,

(iii) $f(0) = \boxed{0}$ and $f(2) = \boxed{\sqrt[3]{2}}$.

Therefore, the maximum and minimum values of f are as follows:

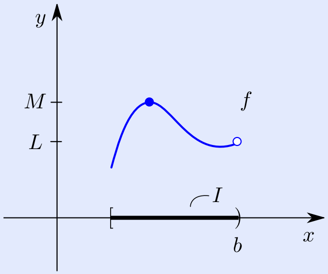
Maximum: $\sqrt[3]{2}$, occurring at $x = 2$,

Minimum: $-\dfrac{1}{\sqrt[3]{4}}$, occurring at $x = 1/2$.

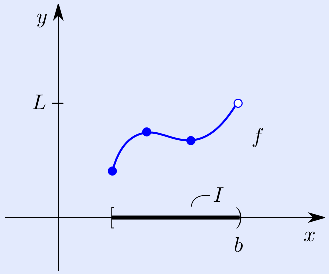
□

28.3. When the interval is not closed

So far we have been considering only closed intervals, that is, intervals that include both endpoints. The method we have been using for finding the extremes of a continuous function on a closed interval can be used even if the interval is not closed, provided we replace the step where we evaluate the function at an endpoint by taking an appropriate limit instead (although the function is no longer guaranteed to have either a maximum or a minimum). There are many possibilities here, but a couple of examples should suffice to give the idea:



maximum M



no maximum

In the diagram on the left, the right-hand endpoint b of the interval is not included, so, instead of evaluating the function there, we compute the limit as x approaches b from the left. Since this limit L is less than the number M we get when evaluating the function where the derivative is zero, it is of no consequence as far as a maximum is concerned, so it can be ignored.

In the diagram on the right, the situation is similar, but here the limit L is greater than numbers we get when evaluating the function at the other candidates (i.e., the heights of the dots). We conclude that the function has no maximum on the interval (the value of the function rises toward L as b is approached but L is never attained).

The analysis is similar if an endpoint is infinite. For instance, if the interval is $[1, \infty)$, then one replaces the step of evaluating the function at the right endpoint by taking the limit of the function as x goes to infinity.

Everything we have said for a maximum value has a corresponding statement for a minimum value.

28.3.1 Example Find the extreme values (if any) of the function

$$f(x) = \frac{3x^2 - 1}{x^2 - 1}$$

on the interval $[-1/2, 1)$ and the x values where they occur. If an extreme value does not exist, explain why not.

Solution We use the quotient rule to find the derivative of f:

$$\begin{aligned} f'(x) &= \frac{\left(x^2-1\right)\frac{d}{dx}\left[3x^2-1\right] - \left(3x^2-1\right)\frac{d}{dx}\left[x^2-1\right]}{\left(x^2-1\right)^2} \\ &= \frac{\left(x^2-1\right)(6x) - \left(3x^2-1\right)(2x)}{\left(x^2-1\right)^2} \\ &= -\frac{4x}{\left(x^2-1\right)^2}. \end{aligned}$$

The derivative is zero when $x = 0$ and it is undefined when the denominator is zero, namely, when $x = \pm 1$. However, since neither 1 nor -1 is in the given interval $[-1/2, 1)$, we omit them from consideration.

Evaluating, we get

(i) $f(0) = \boxed{1}$,

(ii) none,

(iii) $f(-1/2) = \boxed{1/3}$ and for the other end of the interval we compute a limit:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{3x^2-1}{x^2-1} \qquad \left(\frac{\text{about } 2}{\text{small neg.}}\right)$$
$$= \boxed{-\infty} .$$

Therefore,

Maximum: 1, occurring at $x = 0$,

Minimum: none (the value of the function goes to $-\infty$ as x approaches 1 from the left).

□

28.3.2 Example Find the extreme values (if any) of the function

$$f(x) = \frac{x-1}{x^2}$$

on the interval $(0, \infty)$ and the x values where they occur. If an extreme value does not exist, explain why not.

Solution The derivative of f is, using the quotient rule,

$$f'(x) = \frac{x^2(1) - (x-1)(2x)}{x^4} = \frac{2x - x^2}{x^4}.$$

Now f' is zero only when the numerator is zero, that is when $x(2-x) = 2x - x^2 = 0$, giving $x = 0, 2$. However, f' is undefined when $x = 0$ since this value makes the denominator zero, so $x = 0$ gets excluded from the list of values that makes f' zero. In fact, 0 is not even included in the list of values that makes f' undefined since it is not in the given interval $(0, \infty)$. Evaluating we get

(i) $f(2) = \boxed{1/4}$,

(ii) none,

(iii) Here we compute appropriate limits:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \frac{x-1}{x^2} \qquad \left(\frac{\text{about } -1}{\text{small pos.}}\right) \\ &= \boxed{-\infty} \ , \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{x-1}{x^2} = \lim_{x \to \infty} \left(\frac{1}{x} - \frac{1}{x^2}\right) \\ &= \boxed{0} \ . \end{aligned}$$

Therefore,

Maximum: $1/4$, occurring at $x = 2$,

Minimum: none (the value of the function goes to $-\infty$ as x approaches 0 from the right).

$\square$

28 – Exercises

28 – 1 Find the extreme values of the function $f(x) = 4x - x^2$ on the interval $[1, 5]$ and the x values at which they occur. Also, sketch the graph of f.

28 – 2 Find the extreme values of the function $f(x) = x^{2/3}$ on the interval $[-1, 8]$ and the x values at which they occur.

28 – 3 Find the extreme values (if any) of the function

$$f(x) = \ln x + \frac{2}{x}$$

on the interval $(1, e]$ and the x values where they occur. If an extreme value does not exist, explain why not.