3 Exponential and Logarithmic functions

3.1 Exponent

Here are a few examples to remind the reader of the definitions and laws for expressions involving exponents:

$$
\begin{aligned}
2^3 &= 2 \cdot 2 \cdot 2 = 8, \\
2^0 &= 1, \\
2^{-1} &= \frac{1}{2}, \\
2^{-3} &= \frac{1}{2^3} = \frac{1}{8}, \\
9^{1/2} &= \sqrt{9} = 3, \\
9^{3/2} &= (\sqrt{9})^3 = 27 \\
8^{-2/3} &= \frac{1}{8^{2/3}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{4}, \\
a^2 a^3 &= (aa)(aaa) = a^5 = a^{2+3}, \\
(a^2)^3 &= (aa)(aa)(aa) = a^6 = a^{2 \cdot 3}.
\end{aligned}
$$

The last two lines illustrate the following identities, which hold for any real numbers a, x and y with $a > 0$:

Law of exponents.

(i) $a^x a^y = a^{x+y}$,

(ii) $(a^x)^y = a^{xy}$.

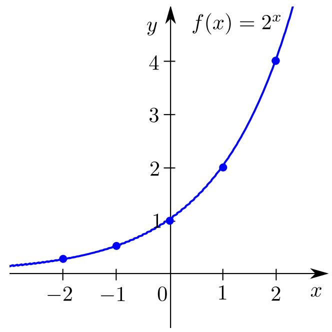
The seemingly strange definitions $2^0 = 1$, $2^{-1} = 1/2$, and so on are forced on us if we wish for the laws of exponents to be valid for all real numbers x and y as stated:

- $2^0 \cdot 2 = 2^0 \cdot 2^1 = 2^{0+1} = 2^1 = 2 \quad \text{forces} \quad 2^0 = 1$,
- $2^{-1} \cdot 2 = 2^{-1} \cdot 2^1 = 2^{(-1)+1} = 2^0 = 1 \quad \text{forces} \quad 2^{-1} = \frac{1}{2}$.

3.2 Exponential function

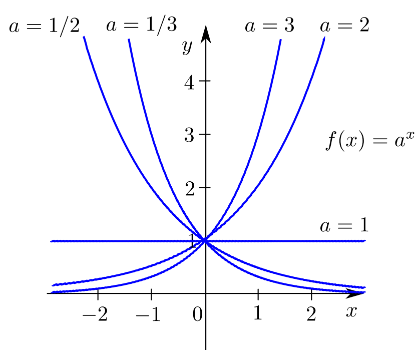
The graph of the function $f(x) = 2^x$ can be sketched by plotting a few points and connecting:

x	$f(x) = 2^x$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4



From the discussion above, we know what is meant by $f(x)$ when x is a rational number (i.e., fraction), since we have $f(m/n) = 2^{m/n} = (\sqrt[n]{2})^m$. But since we have not defined 2^x when x is irrational, the graph above actually has a hole in it above every irrational number, such as $\sqrt{3}$. We remedy this situation by defining 2^x for irrational x in such a way that the graph is smooth. (This informal way of extending f to the irrational numbers will be suitable for our purposes. A rigorous treatment requires the notion of a limit.)

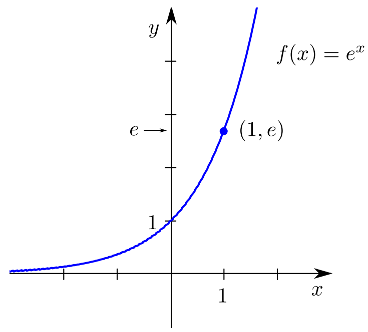
For any positive real number a, we have an **exponential function** f given by $f(x) = a^x$. Here are the graphs of f for a few choices of a:



In general, the graph falls from left to right if $a < 1$ and it rises from left to right if $a > 1$.

The base e. The most common choice for the base a is a certain number $e \approx 2.71828$. The reason for this choice is that it allows simplification of several formulas involving exponential functions. The function $f(x) = e^x$ is often called *the* exponential function.

Since $e > 1$, the graph of this exponential function rises from left to right. It passes through the y-axis at 1 (as do the graphs of all the exponential functions), and it passes through the point $(1, e)$:



3.3 Logarithm

The notation $\log_2 8$ can be thought of as posing a question: What power do you raise 2 to in order to get 8? In symbols,

$$\log_2 8 = ? \qquad \Longleftrightarrow \qquad 2^? = 8.$$

The answer is 3, that is, $\log_2 8 = 3$.

In general, for any positive real numbers a and y with $a \neq 1$, the notation $\log_a y$ stands for the power that a is raised to in order to get y; it is the **logarithm** with respect to the base a of y (or just "log base a of y") :

$$\log_a y = x \qquad \Longleftrightarrow \qquad a^x = y. \tag{1}$$

From this relationship between the logarithm and the exponential and the laws of exponents, we get the following identities:

LAWS OF LOGARITHMS. For any positive real numbers x and y and any real number r,

(i) $\log_a(xy) = \log_a x + \log_a y$,

(ii) $\log_a(x/y) = \log_a x - \log_a y$,

(iii) $\log_a x^r = r \log_a x$,

3.3.1 Example Express $\log_5 72$ in terms of $\log_5 2$ and $\log_5 3$.

Solution We have

$$\log_5 72 = \log_5(2^3 3^2) = \log_5 2^3 + \log_5 3^2 = 3\log_5 2 + 2\log_5 3.$$

□

The following identities follow immediately from (1):

Inverse properties of logarithm.

(i) $\log_a a^x = x \qquad (x \in \mathbf{R})$,

(ii) $a^{\log_a x} = x \qquad (x > 0)$.

These identities can be used to solve equations involving exponentials or logarithms as the next example shows.

3.3.2 Example

(a) Solve $3^{2x+1} = 4$ for x.

(b) Solve $\log_5(1 - x) = 2$ for x.

Solution

(a) Applying $\log_3$ to both sides and then using the identity (i) above, we have

$$\begin{aligned} 3^{2x+1} = 4 \qquad &\Longrightarrow \qquad \log_3 3^{2x+1} = \log_3 4 \\ &\Longrightarrow \qquad 2x + 1 = \log_3 4 \\ &\Longrightarrow \qquad x = \frac{1}{2}(\log_3 4 - 1) \end{aligned}$$

(b) Raising 5 to both sides and then using the identity (ii), we have

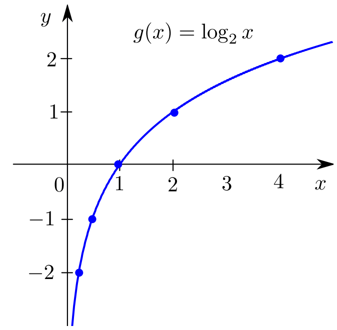
$$\begin{aligned} \log_5(1 - x) = 2 \qquad &\Longrightarrow \qquad 5^{\log_5(1-x)} = 5^2 \\ &\Longrightarrow \qquad 1 - x = 25 \\ &\Longrightarrow \qquad x = -24 \end{aligned}$$

□

3.4 Logarithmic function

The graph of the function $g(x) = \log_2 x$ can be sketched by plotting a few points and connecting:

x	$g(x) = \log_2 x$
1/4	−2
1/2	−1
1	0
2	1
4	2



We have used for inputs only positive numbers since $\log_2 x$ is not defined when x is ≤ 0 (there is nothing you can raise 2 to in order to get a number ≤ 0).

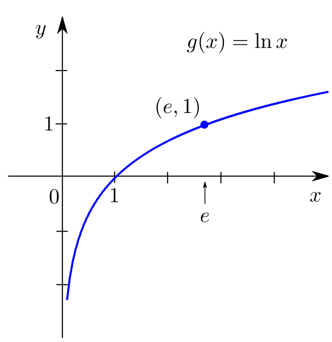
This graph is the reflection across the 45° line $y = x$ of the graph of $f(x) = 2^x$ (see 3.2). We know from 2.5 that the graphs of a function and its inverse are always related in this way, so we might suppose that g is the inverse of f. This is in fact the case, since, using (1) of 3.3, we get

$$g(y) = x \quad \Longleftrightarrow \quad \log_2 y = x \quad \Longleftrightarrow \quad 2^x = y \quad \Longleftrightarrow \quad f(x) = y,$$

so g is f^{-1} (see (1) of 2.5).

More generally, for any positive real number a, the **logarithmic function** $g(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$, so the graphs of these two functions are related as just described. In particular, if $a > 1$, then the graph of g rises from left to right; if $0 < a < 1$, it falls from left to right (cf. 3.2).

Natural logarithmic function. Of particular importance for us is the natural logarithmic function $g(x) = \ln x$, where ln means $\log_e$. Its graph is the reflection across the 45° line $y = x$ of the graph of the exponential function $f(x) = e^x$:



3 – Exercises

3 – 1 Simplify the following expressions:

(a) $16^{-3/4}$

(b) $8^{1/6}8^{3/2}$

(c) $\log_3 \frac{1}{27}$

(d) $e^{4 \ln 3}$

3 – 2 Sketch the graph of the equation $y = 2 - e^{x+1}$ in steps:

(a) Sketch the graph of $y = e^{x+1}$. (Hint: The graph is a shifted version of the graph of $y = e^x$ shown in 3.2.)

(b) Sketch the graph of $y = -e^{x+1}$. (Hint: The graph is a reflected version of the graph of $y = e^{x+1}$.)

(c) Sketch the graph of $y = 2 - e^{x+1}$. (Hint: The graph is a shifted version of the graph of $y = -e^{x+1}$.)

3 – 3

(a) Solve $e^{1-x^2} = 2$ for x.

(b) Solve $\log_5(x - 4) + \log_5 x = 1$ for x.

3 – 4 Show that any logarithm can be expressed in terms of natural logarithms by establishing the formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

Hint: Write $y = \log_a x$ in the corresponding exponential form, apply ln to both sides, and solve for y.

3 – 5 Suppose that the expression $2^{1/3}$ has not yet been defined. Give an argument similar to those given at the end of Section 3.1 to show that $2^{1/3}$ is forced by the laws of exponents to equal $\sqrt[3]{2}$.