25. Summary of derivative rules

25.1. Tables

The derivative rules that have been presented in the last several sections are collected together in the following tables. The first table gives the derivatives of the basic functions; the second table gives the rules that express a derivative of a function in terms of the derivatives of its component parts (the "derivative decomposition rules"). For the sake of completeness, a few new rules have been added.

Derivatives of basic functions.

$$\frac{d}{dx}[c] = 0 \quad (c, \text{ constant}) \qquad \frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[a^x] = a^x \ln a$$

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \qquad \frac{d}{dx}[\log_a|x|] = \frac{1}{x \ln a}$$

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{x\sqrt{x^2-1}}$$

The rules for the derivative of a logarithm have been extended to handle the case of $x < 0$ by the addition of absolute value signs. If the absolute value signs are removed, the rules are still valid, but only for $x > 0$.

Derivative decomposition rules.

Constant multiple rule

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$$

Sum/Difference rule

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x) + f(x)\frac{d}{dx}\left[g(x)\right]$$

Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{(g(x))^2}$$

Chain rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

25.2. Examples

25.2.1 Example Verify the rule $\frac{d}{dx}[\tan x] = \sec^2 x$.

Solution

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \\ &= \frac{\cos x \frac{d}{dx}[\sin x] - \sin x \frac{d}{dx}[\cos x]}{(\cos x)^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

□

25.2.2 Example Find the derivative of $f(x) = \sqrt[3]{\tan 5x + 1}$.

Solution

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\sqrt[3]{\tan 5x + 1}\right] \\
&= \tfrac{1}{3}(\tan 5x + 1)^{-2/3} \cdot \frac{d}{dx}[\tan 5x + 1] \\
&= \tfrac{1}{3}(\tan 5x + 1)^{-2/3} \cdot \sec^2 5x \cdot 5 \\
&= \frac{5\sec^2 5x}{3(\sqrt[3]{\tan 5x + 1})^2}.
\end{aligned}
$$

□

25.2.3 Example Find the derivatives of each of the following functions. Avoid the quotient rule.

(a) $f(x) = \dfrac{\sin 3x}{4}$

(b) $f(x) = \dfrac{4}{\sin 3x}$

Solution When either the numerator or the denominator is constant, the quotient rule can be avoided by first rewriting, and such a solution is generally easier than one using the quotient rule. (If neither numerator nor denominator is constant, then rewriting in order to use the product rule requires more steps than just using the quotient rule so is not advised.)

(a) Rewriting as $f(x) = \frac{1}{4}\sin 3x$, we have $f'(x) = \frac{1}{4}\cos 3x \cdot 3$.

(b) Rewriting as $f(x) = 4(\sin 3x)^{-1}$, we have

$$f'(x) = -4(\sin 3x)^{-2} \cdot \cos 3x \cdot 3 = -\frac{12\cos 3x}{\sin^2 3x}.$$

□

25.2.4 Example Find the derivative of $y = \dfrac{7}{\tan^{-1} 2x}$.

Solution We first rewrite as $y = 7(\tan^{-1} 2x)^{-1}$ to avoid using the quotient rule. (Incidentally, the -1's do not cancel since $\tan^{-1} 2x$ denotes the inverse tangent of $2x$ rather than $(\tan 2x)^{-1}$.) We have

$$y' = -7(\tan^{-1} 2x)^{-2} \cdot \frac{1}{1+(2x)^2} \cdot 2 = -\frac{14}{(\tan^{-1} 2x)^2(1+4x^2)}.$$

□

25.2.5 Example Differentiate $f(t) = t^3 e^{\sec 2t}$.

Solution

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left[t^3 e^{\sec 2t}\right] \\ &= \frac{d}{dt}\left[t^3\right] e^{\sec 2t} + t^3 \frac{d}{dt}\left[e^{\sec 2t}\right] \\ &= 3t^2 e^{\sec 2t} + t^3 e^{\sec 2t} \frac{d}{dt}[\sec 2t] \\ &= 3t^2 e^{\sec 2t} + t^3 e^{\sec 2t} \sec 2t \tan 2t \cdot 2 \\ &= e^{\sec 2t}\left(3t^2 + 2t^3 \sec 2t \tan 2t\right). \end{aligned}$$

□

25.2.6 Example Find the derivative of $f(x) = \cot^x x$.

Solution Since $\cot^x x$ means $(\cot x)^x$, this is a case where neither base nor exponent is constant, so logarithmic differentiation is required:

$$\begin{aligned} \ln(f(x)) &= \ln \cot^x x \\ &= x \ln \cot x, \end{aligned}$$

so

$$\begin{aligned} \frac{d}{dx}\left[\ln(f(x))\right] &= \frac{d}{dx}\left[x \ln \cot x\right] \\ \frac{1}{f(x)} \cdot f'(x) &= \frac{d}{dx}[x] \ln \cot x + x \frac{d}{dx}[\ln \cot x] \\ &= \ln \cot x + x \frac{1}{\cot x} \frac{d}{dx}[\cot x] \\ &= \ln \cot x + x \frac{1}{\cot x}(-\csc^2 x) \\ &= \ln \cot x - \frac{x}{\cos x \sin x}, \end{aligned}$$

and finally

$$\begin{aligned} f'(x) &= f(x)\left(\ln \cot x - \frac{x}{\cos x \sin x}\right) \\ &= \cot^x x \left(\ln \cot x - \frac{x}{\cos x \sin x}\right). \end{aligned}$$

□

25.2.7 Example Given $y = \csc^3\left(\dfrac{5^{2x}+x}{x\sin 3x - \ln(4+x^2)}\right)$, find y' without using intermediate steps.

Solution

$$\begin{aligned} y' = 3\csc^2&\left(\frac{5^{2x}+x}{x\sin 3x-\ln(4+x^2)}\right)\\ &\cdot -\csc\left(\frac{5^{2x}+x}{x\sin 3x-\ln(4+x^2)}\right)\cot\left(\frac{5^{2x}+x}{x\sin 3x-\ln(4+x^2)}\right)\\ &\cdot \frac{\left(x\sin 3x-\ln\left(4+x^2\right)\right)\left(5^{2x}\ln 5(2)+1\right)-\left(5^{2x}+x\right)\left(\sin 3x+3x\cos 3x-\frac{2x}{4+x^2}\right)}{\left(x\sin 3x-\ln(4+x^2)\right)^2}. \end{aligned}$$

(We used, in order, (1) the chain rule with the cubing function as the outside function, (2) the chain rule with the cosecant function as the outside function, (3) the quotient rule.) □

25 – Exercises

25 – 1 Verify the rule $\frac{d}{dx}[\sec x] = \sec x \tan x$.

25 – 2 Find the derivative of each of the following functions:

(a) $f(x) = \sqrt[5]{4x - \sec 3x}$.

(b) $f(t) = \tan^{-1} e^{5t}$.

25 – 3 Find the derivative of each of the following functions:

(a) $y = 2^{\csc 7x} \tan(\ln 4x)$.

(b) $y = \sec^{-1}\left(\frac{e^{3t}}{1+t^2}\right)$.

25 – 4 Differentiate $f(x) = \tan^{e^x} x$.

25 – 5 Find the derivative of the following function without using intermediate steps:

$$f(x) = \sin^6\left(\frac{e^{\sec 3x}}{4x^3 - 2x + 5}\right).$$