22. Derivative of inverse function

22.1. Statement

Any time we have a function f, it makes sense to form is inverse function f^{-1} (although this often requires a reduction in the domain of f in order to make it injective). If we know the derivative of f, then we can find the derivative of f^{-1} as follows:

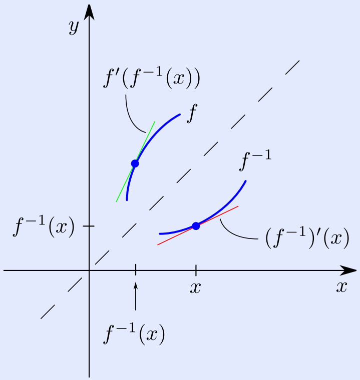
Derivative of inverse function. If f is a function with inverse function f^{-1}, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This formula is an immediate consequence of the definition of an inverse function and the chain rule:

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{d}{dx}\left[f(f^{-1}(x))\right] &= \frac{d}{dx}[x] \\ f'(f^{-1}(x)) \cdot (f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}. \end{aligned}$$

The figure below shows that the formula agrees with the fact that the graph of f^{-1} is the reflection across the 45° line $y = x$ of the graph of f. Such a reflection interchanges the coordinates of a point (i.e., (x, y) reflects to (y, x)), so the reflection of a line has slope the reciprocal of the slope of the original line. Thus, the slope of the line tangent to the graph of f^{-1} at the point $(x, f^{-1}(x))$ (red line) is the reciprocal of the slope of the tangent to the graph of f at the point $(f^{-1}(x), x)$ (green line), and this is also what the formula says.



22.1.1 Example The inverse of the function $f(x) = x^2$ with reduced domain $[0, \infty)$ is $f^{-1}(x) = \sqrt{x}$. Use the formula given above to find the derivative of f^{-1}.

Solution We have $f'(x) = 2x$, so that $f'(f^{-1}(x)) = 2\sqrt{x}$. Using the formula above, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2\sqrt{x}}.$$

(We can check by using the power rule:

$$(f^{-1})'(x) = \frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}},$$

in agreement with what we just found.) □

22.2. Derivative of logarithm function

The logarithm function $\log_a x$ is the inverse of the exponential function a^x. Therefore, we can use the formula from the previous section to obtain its derivative.

> Derivative of logarithm function. For any positive real number a,
>
> $$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}.$$
>
> In particular,
>
> $$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

The second formula follows from the first since $\ln e = 1$. We verify the first formula. The function $f(x) = a^x$ has inverse function $f^{-1}(x) = \log_a x$. We have $f'(x) = a^x \ln a$, so $f'(f^{-1}(x)) = a^{\log_a x}\ln a = x\ln a$. Using the formula for the derivative of an inverse function, we get

$$\frac{d}{dx}[\log_a x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{x\ln a},$$

as claimed.

22.2.1 Example Find the derivative of each of the following functions:

(a) $f(x) = 4\log_2 x + 5x^3$

(b) $f(x) = \ln(\sin x)$

Solution

(a) Using the new rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[4\log_2 x + 5x^3\right] \\ &= 4\frac{d}{dx}\left[\log_2 x\right] + \frac{d}{dx}\left[5x^3\right] \\ &= 4\left(\frac{1}{x\ln 2}\right) + 15x^2 \\ &= \frac{4}{x\ln 2} + 15x^2. \end{aligned}$$

(b) Here, we require the chain rule (with outside function the natural logarithm):

$$f'(x) = \frac{d}{dx}\left[\ln(\sin x)\right] = \frac{1}{\sin x}\cdot\cos x = \cot x.$$

□

22.3. Derivatives of inverse sine and inverse cosine functions

The formula for the derivative of an inverse function can be used to obtain the following derivative formulas for $\sin^{-1} x$ and $\cos^{-1} x$:

Derivatives of inverse sine and inverse cosine functions.

(i) $\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1-x^2}}$,

(ii) $\dfrac{d}{dx}\left[\cos^{-1} x\right] = -\dfrac{1}{\sqrt{1-x^2}}$,

We verify the first formula. The function $f(x) = \sin x$ with domain reduced to $[-\pi/2, \pi/2]$ has inverse function $f^{-1}(x) = \sin^{-1} x$. We have $f'(x) = \cos x$, so that $f'(f^{-1}(x)) = \cos\left(\sin^{-1}(x)\right)$. The formula for the derivative of an inverse function now gives

$$\frac{d}{dx}\left[\sin^{-1} x\right] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos\left(\sin^{-1}(x)\right)}.$$

This last expression can be simplified by using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. Put $\theta = \sin^{-1}(x)$ and note that $\theta \in [-\pi/2, \pi/2]$. Then,

$$\cos\left(\sin^{-1}(x)\right) = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2},$$

where we have used that $\cos\theta \geq 0$ in choosing the positive square root when we solved the trigonometric identity for $\cos\theta$. Putting this final expression into the earlier equation, we get

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}},$$

as claimed.

22.3.1 Example Find the derivative of each of the following functions:

(a) $f(x) = 4e^x \sin^{-1} x$,

(b) $f(x) = \cos^{-1}\left(x^3 + x\right)$.

Solution

(a) The product rule is applied first:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[4e^x \sin^{-1} x\right] \\ &= \frac{d}{dx}\left[4e^x\right] \sin^{-1} x + 4e^x \frac{d}{dx}\left[\sin^{-1} x\right] \\ &= (4e^x) \sin^{-1} x + 4e^x \left(\frac{1}{\sqrt{1-x^2}}\right) \\ &= 4e^x \sin^{-1} x + \frac{4e^x}{\sqrt{1-x^2}}. \end{aligned}$$

(b) The chain rule is used with outside function the arc cosine function:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\cos^{-1}\left(x^3 + x\right)\right] \\ &= -\frac{1}{\sqrt{1-(x^3+x)^2}} \cdot (3x^2 + 1) \\ &= -\frac{3x^2+1}{\sqrt{1-(x^3+x)^2}}. \end{aligned}$$

□

22 – Exercises

22 – 1 Find the derivatives of each of the following functions:

(a) $f(x) = 3\log_3 x - 4\ln x$,

(b) $f(t) = \ln\left(1 + 3e^{2t}\right)$.

22 – 2 Find the derivative of the function $f(x) = \ln(\ln(\ln x))$.

22 – 3 Find the derivative of the function $f(x) = \sin^{-1}\left(\sqrt{1 - x^2}\right)$.