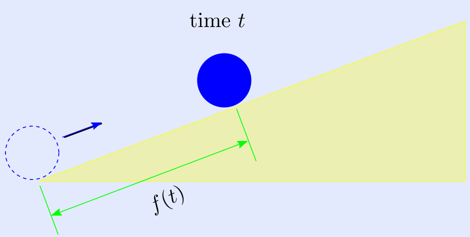
14. Derivative as rate of change

In the last section, we focused on the geometrical interpretation of the derivative as a general slope function. Here, we see that the derivative can be interpreted as a rate function.

14.1. Rolling ball

A ball is given an initial push up an incline. After t seconds, its position in meters is $f(t)$, where $f(t) = 8t - t^2$.



Here is a table of position versus time:

$f(t)$ (m)	0	7	12	15	16	15	12	7	0
t (s)	0	1	2	3	4	5	6	7	8

The ball slows down during the first four seconds (traveling 7 m during the first second, 5 m during the second, 3 m during the third, and 1 m during the fourth). It comes to a standstill right at 4 s having traveled 16 m up the incline. It then rolls back down the incline, reaching the initial position 0 m at time 8 s.

The ball's average velocity $\bar{v}$ between two times t_1 and t_2 is the difference of the ball's positions at these times divided by the difference of the times:

$$\bar{v} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}. \tag{1}$$

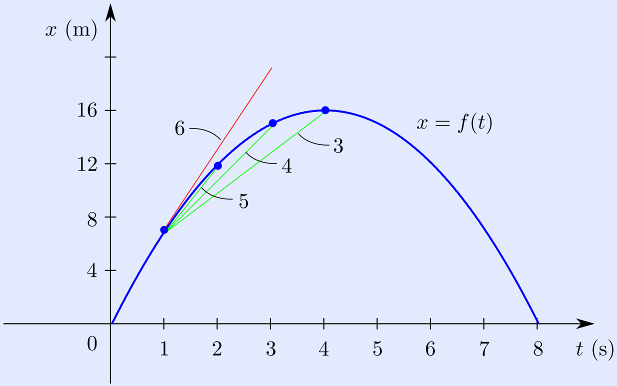
For instance, between the two times 1 and 3, the ball's average velocity is

$$\bar{v} = \frac{f(3) - f(1)}{3 - 1} = \frac{15 - 7}{3 - 1} = 4 \text{ m/s}.$$

This average velocity is the constant velocity that the ball would have to maintain over the time interval $[1, 3]$ in order to end at the same position it ends at with its given variable velocity. (Check: At the constant velocity 4 m/s, the ball, starting at 7 m, would travel $(4 \text{ m/s}) \cdot (2 \text{ s}) = 8$ m over the 2 s interval $[1, 3]$, putting it at 15 m as claimed.)

Taking t_1 to be 1 each time and taking for t_2 some times that get closer and closer to 1 we get the following table of average velocities:

t_1	t_2	$\bar{v}$ (m/s)
1	4	3
1	3	4
1	2	5
1	1.5	5.5
1	1.25	5.75



As can be seen from the definition (1), average velocity is the slope of the secant line through the two points on the graph of f corresponding to the times t_1 and t_2. Shown above is the graph of f and the secant lines corresponding to the first three entries in the table (each tagged with its slope).

We are interested in the velocity of the ball right at the time 1 (by which we mean what the ball's speedometer, if it had one, would read at the time 1). The average velocities with $t_1 = 1$ approximate the velocity at 1 and those with t_2 ever closer to 1 provide ever better approximations (since shorter lapses of time allow for less fluctuation in velocity). Therefore, letting h be the elapsed time between 1 and t_2 (so that $t_2 = 1 + h$) we have

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$$
\begin{array}{cccc}
v(1) & = & \lim\limits_{h\to 0} & \dfrac{f(1+h)-f(1)}{h}. \\
\uparrow & & \uparrow & \uparrow \\
\text{velocity} & & \text{ever shorter} & \text{average velocity} \\
\text{at time 1} & & \text{elapsed time} & \text{between 1 and } 1+h
\end{array}
$$

The right-hand side of this equation is the derivative of the position function f evaluated at 1; in symbols, $v(1) = f'(1)$. Therefore, the velocity at time 1 is the slope of the line tangent to the graph of f at the point $(1, f(1))$. It is shown in the example below that this slope is 6. The tangent line appears in red in the figure above, tagged with its slope 6. The slopes of the secant lines (average velocities) approach this slope as the elapsed times between 1 and t_2 get smaller as expected.

More generally, the velocity $v(t)$ of the ball at any time t is the derivative of the position function at time t:

$$v(t) = f'(t).$$

14.1.1 Example Find the velocity of the ball at each second, starting at 0 and ending at 8.

Solution The velocity at time t is $v(t) = f'(t)$, so, using the definition of the derivative,

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we have

$$\begin{aligned}
v(t) = f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h\to 0} \frac{(8(t+h) - (t+h)^2) - (8t - t^2))}{h} \\
&= \lim_{h\to 0} \frac{(8t + 8h - (t^2 + 2th + h^2)) - 8t + t^2}{h} \\
&= \lim_{h\to 0} \frac{(8 - 2t - h)h}{h} = \lim_{h\to 0} 8 - 2t - h \\
&= 8 - 2t.
\end{aligned}$$

Therefore, the desired velocities are as indicated in the following table:

$v(t)$ (m/s)	8	6	4	2	0	−2	−4	−6	−8
t (s)	0	1	2	3	4	5	6	7	8

(The negative velocities in the table indicate that position decreases with time consistent with the fact that the ball rolls back down the incline after 4 s.) □

14.2. Rate of change

Velocity, as discussed above in the example of the rolling ball, is an example of a rate of change. It is the rate at which position $f(t)$ is changing with respect to time t. The general notion of rate of change is as follows:

DEFINITION. Let f be a function. The **average rate of change** of f between x_1 and x_2 is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The (**instantaneous**) **rate of change** of f at a is the derivative of f evaluated at a, that is, $f'(a)$.

14.2.1 Example A wind farm selling electricity at a price of x cents per kilowatt hour (¢/kWh) has found that its profit P ($1000/yr) depends on the price x according to the equation

$$P(x) = -x^2 + 10x - 2.$$

(a) Find the wind farm's profit per year when the price is 2 ¢/kWh, and also when the price is 5 ¢/kWh.

(b) Find the average rate of change of profit if the price is changed from 2 ¢/kWh to 5 ¢/kWh and give an interpretation.

(c) Find the (instantaneous) rate of change of profit at a price of 2 ¢/kWh.

Solution

(a) At a price of 2, the wind farm's profit is

$$P(2) = -(2)^2 + 10(2) - 2 = 14,$$

that is, $14,000/yr. Similarly, at a price of 5, the profit is $P(5) = 23$, that is, $23,000/yr.

(b) The average rate of change of profit is

$$\frac{\text{change in profit}}{\text{change in price}} = \frac{P(5) - P(2)}{5 - 2} = \frac{23 - 14}{3} = 3.$$

The units here are profit units (\$1,000/yr) over price units (¢/kWh) so the answer is \$3,000/yr per ¢/kWh. The interpretation is that for every cent per kilowatt hour the price is increased, the wind farm's profit increases by \$3,000 per year on average.

(c) The (instantaneous) rate of change of profit at the price 2 is

$$\begin{aligned} P'(2) &= \lim_{h \to 0} \frac{P(2+h) - P(2)}{h} \\ &= \lim_{h \to 0} \frac{(-(2+h)^2 + 10(2+h) - 2) - 14}{h} \\ &= \lim_{h \to 0} \frac{-(4 + 4h + h^2) + 20 + 10h - 16}{h} \\ &= \lim_{h \to 0} \frac{6h - h^2}{h} = \lim_{h \to 0} 6 - h \\ &= 6, \end{aligned}$$

that is, \$6,000/yr per ¢/kWh.

□

14 – Exercises

14 – 1 A ball, dropped from a height of 100 m, has height $f(t) = 100 - 10t^2$ (in meters) after t seconds.

(a) Find the ball's average velocity between the times 0 s and 3 s.

(b) Find the ball's velocity at the time 1 s.

(c) Find the time at which the ball's velocity is 50 m/s (downward).

(d) Find the ball's velocity at the time of impact with the ground.

14 – 2 The spring quail population in Conecuh National Forest depends on the amount of feed placed throughout the forest during the preceding winter. A spring census finds that the population is

$$f(x) = \frac{2x+3}{x+2}$$

in thousands of individuals when the feed is x (hundreds of kilograms).

(a) Find the average rate of change of population between the feed amounts 400 kg and 600 kg and give an interpretation.

(b) Find the rate of change of population when the feed amount is 400 kg.