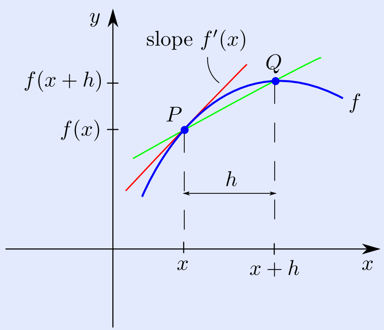
13. Definition of derivative

13.1. Definition and geometrical interpretation

The goal here is to express the slope of the line tangent to the graph of a function. In the next section, we will see that this slope gives the (instantaneous) rate of change of the function.

Pictured below is the graph of a function f. We would like to find the slope of the line tangent to the graph of f at the point P (red line). However, in order to find the slope of a line, we need two points on the line (so that we can take the difference of the y-coordinates over the difference of the x-coordinates). We could use P as one of the points, but there is no obvious way to come up with a second point.



If we push to the side a small distance h, we get a second point Q on the graph. The line through P and Q is called a *secant line* (green line). We can find the slope of the secant line by using the two points P and Q:

$$\text{slope of secant} = \frac{f(x+h) - f(x)}{h}.$$

We can think of the secant line as being an approximation to the tangent line. This approximation becomes better and better the closer Q is to P, that is, the smaller h is. Therefore, the slope of the tangent line, denoted $f'(x)$, is the limit of the slope of the secant line as h approaches 0:

$$\begin{array}{ccccc} f'(x) & = & \displaystyle\lim_{h \to 0} & \displaystyle\frac{f(x+h) - f(x)}{h}. \\ \uparrow & & \uparrow & \uparrow \\ \text{slope of} & & \text{slide } Q & \text{slope of} \\ \text{tangent} & & \text{toward } P & \text{secant} \end{array}$$

Since x can be any number (for which the limit exists), the formula above defines a function f', called the derivative of f.

> **Definition of derivative.** The **derivative** of the function f is the function f' defined by
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

An x for which the above limit does not exist is not in the domain of f'.

The derivative f' is regarded as a general slope function. It can be used to find the slope of any line tangent to the graph of f: If P is a point on the graph, then the slope of the tangent line at P is obtained by evaluating the derivative f' at the x-coordinate of P.

13.2. Finding derivative directly from definition

The student who has had some calculus before might know some rules for finding a derivative that allow one to avoid evaluating a limit. We will eventually obtain these rules. However, for the time being we will be finding the derivative of a function f by using the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This is called "finding the derivative directly from the definition."

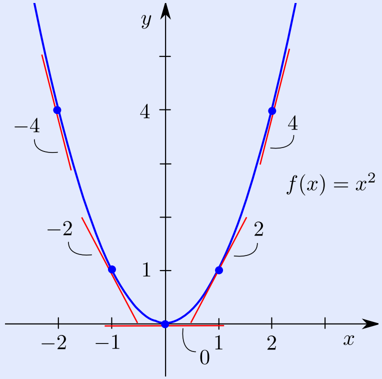
13.2.1 Example Find the derivative of $f(x) = x^2$ directly from the definition, use it to find the slopes of the lines tangent to the curve at the points with x-coordinates $x = -2, -1, 0, 1, 2$, and sketch the graph of f together with these tangent lines.

Solution We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} 2x + h \\ &= 2x. \end{aligned}$$

The derivative $f'(x) = 2x$ is a general slope function. The slopes of the tangent lines are obtained by evaluating the derivative at the given values of x (see table). The graph is shown with the tangent lines tagged with their slopes.

$f'(x)$	-4	-2	0	2	4
x	-2	-1	0	1	2



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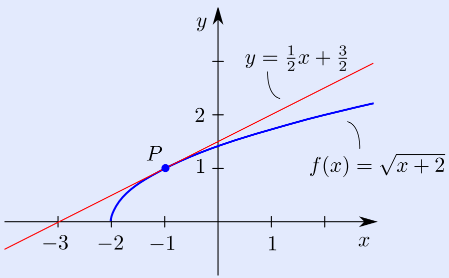
13.2.2 Example Find the derivative of $f(x) = \sqrt{x+2}$ directly from the definition, use it to find an equation of the line tangent to the graph of f at the point $P(-1, 1)$, and sketch the graph of f together with this tangent line.

Solution We have (using the rationalization method in the process)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{(x+h)+2}+\sqrt{x+2}}{\sqrt{(x+h)+2}+\sqrt{x+2}}\\
&= \lim_{h\to 0}\frac{((x+h)+2)-(x+2)}{h(\sqrt{(x+h)+2}+\sqrt{x+2})}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{(x+h)+2}+\sqrt{x+2})}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{(x+h)+2}+\sqrt{x+2}}\\
&= \frac{1}{2\sqrt{x+2}}.
\end{aligned}
$$

The slope of the line tangent to the graph of f at $P(-1,1)$ is $m = f'(-1) = \frac{1}{2}$, so this tangent line has equation $y-1 = \frac{1}{2}(x-(-1))$, which has slope-intercept form $y = \frac{1}{2}x+\frac{3}{2}$. The sketch is



13.2.3 Example Find the derivative of $f(x) = \dfrac{x+1}{x-1}$ directly from the definition.

Solution We have (using the combining fractions method in the process)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)+1}{(x+h)-1} - \dfrac{x+1}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h+1)(x-1)}{(x+h-1)(x-1)} - \dfrac{(x+h-1)(x+1)}{(x+h-1)(x-1)}}{h} \\
&= \lim_{h\to 0} \frac{(x+h+1)(x-1)-(x+h-1)(x+1)}{(x+h-1)(x-1)h} \\
&= \lim_{h\to 0} \frac{(x^2-x+hx-h+x-1)-(x^2+x+hx+h-x-1)}{(x+h-1)(x-1)h} \\
&= \lim_{h\to 0} \frac{-2h}{(x+h-1)(x-1)h} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} \\
&= \frac{-2}{(x-1)^2}.
\end{aligned}
$$

□

The act of finding the derivative of a function is called **differentiation**. For instance, instead of saying "Find the derivative of the function $f(x) = x^2$," one could say "Differentiate the function $f(x) = x^2$."

13 – Exercises

13 – 1 Find the derivative of $f(x) = -x^2 + 2x - 1$ directly from the definition, use it to find the slopes of the lines tangent to the curve at the points with x-coordinates $x = -1, 0, 1, 2, 3$, and sketch the graph of f together with these tangent lines.

13 – 2 Let $f(x) = \dfrac{1}{x^2}$.

(a) Find the derivative of f directly from the definition.

(b) Find the x-intercept of the line tangent to the graph of f at the point $(1, 1)$.

13 – 3 Find the derivative of $f(x) = \sqrt{x}$ directly from the definition.