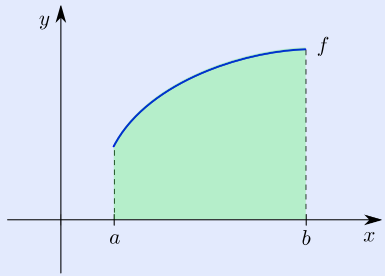
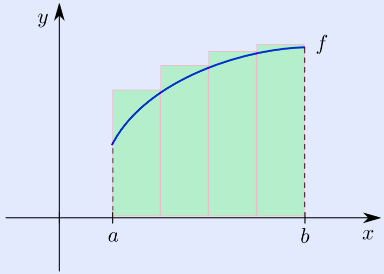
36. Definite integral

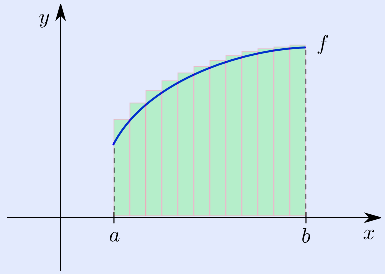
36.1. Rectangle approximation

The area of an odd-shaped region such as that pictured here is not given by a formula from elementary geometry:



However, this area can be approximated by summing rectangle areas (left diagram below), and the more rectangles that are used, the better the approximation becomes (right diagram below):





36.1.1 Example Approximate the area A of the region between the graph of $f(x) = x^2 + 1$ and the x-axis and between $x = 0$ and $x = 6$ as follows:

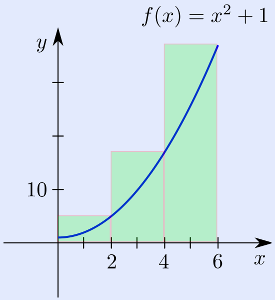
(a) Use three rectangles with the height of each rectangle given by the height of the graph above the right endpoint of the rectangle's base. Do the same using left endpoints.

(b) Repeat part (a) using six rectangles.

Solution

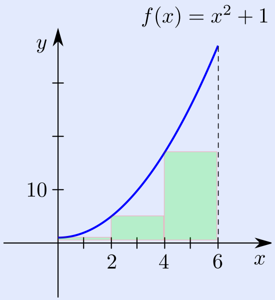
(a) Using right endpoints:

$$
\begin{aligned}
A &\approx f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 \\
&= 5\cdot 2 + 17\cdot 2 + 37\cdot 2 \\
&= 118
\end{aligned}
$$



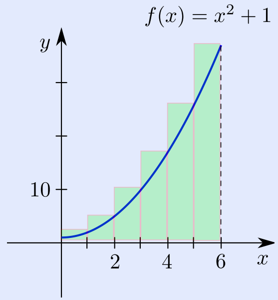
Using left endpoints:

$$
\begin{aligned}
A &\approx f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 \\
&= 1\cdot 2 + 5\cdot 2 + 17\cdot 2 \\
&= 46
\end{aligned}
$$



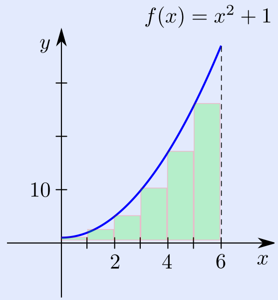
(b) Using right endpoints:

$$
\begin{aligned}
A &\approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 \\
&\qquad + f(4)\cdot 1 + f(5)\cdot 1 + f(6)\cdot 1 \\
&= 2\cdot 1 + 5\cdot 1 + 10\cdot 1 \\
&\qquad + 17\cdot 1 + 26\cdot 1 + 37\cdot 1 \\
&= 97
\end{aligned}
$$



Using left endpoints:

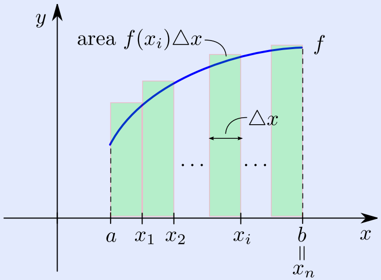
$$
\begin{aligned}
A &\approx f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 \\
&\qquad + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 \\
&= 1\cdot 1 + 2\cdot 1 + 5\cdot 1 \\
&\qquad + 10\cdot 1 + 17\cdot 1 + 26\cdot 1 \\
&= 61
\end{aligned}
$$



□

36.2. Definition of definite integral

If we can write an expression for the sum of the areas of not three rectangles, not six rectangles, but n rectangles for any positive integer n, then the limit of that expression as $n \to \infty$ (i.e., as the number of rectangles goes to infinity) should give the exact area under the curve. Here is a diagram of the general situation:



The sum of the n rectangle areas is

$$\begin{aligned} A_n &= f(x_1)\triangle x + f(x_2)\triangle x + \cdots + f(x_n)\triangle x \\ &= \sum_{i=1}^{n} f(x_i)\triangle x. \end{aligned}$$

This last line is a shorthand notation for the preceding line. It is referred to as summation notation and is read "the sum as i goes from 1 to n of $f(x_i)\triangle x$." The meaning is that one

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should replace i in the expression by 1, then 2, then 3, and so forth all the way up to n, and add the results.

We need a few facts about the summation notation.

SUMMATION NOTATION IDENTITIES.

(i) $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(ii) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \qquad (c,\ \text{constant})$

(iii) $\sum_{i=1}^{n} c = nc \qquad (c,\ \text{constant})$

(iv) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(v) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

The first three of these identities follow from familiar properties of addition and multiplication and can be verified by writing the expressions out in long hand (for instance, the second property is the distributive law). The last two properties are verified by using mathematical induction.

36.2.1 Example

(a) Find a formula for A_n using $f(x) = x^2 + 1$ and the interval $[0, 6]$ and check that what it gives for the cases $n = 3$ (three rectangles) and $n = 6$ (six rectangles) agree with those in Example 36.1.1.

(b) Find the exact area A of the region between the graph of f and the x-axis from $x = 0$ to $x = 6$.

Solution

(a) In the formula for A_n given above $\triangle x = 6/n$ (the interval $[0, 6]$ is divided into n equal parts). Also, $x_1 = \triangle x$, $x_2 = 2\triangle x$, and in general $x_i = i\triangle x = (6i)/n$. Therefore, using the summation notation identities we get

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\triangle x = \sum_{i=1}^{n} \left(x_i^2 + 1\right)\triangle x \\
&= \sum_{i=1}^{n} \left(\left(\frac{6i}{n}\right)^2 + 1\right)\frac{6}{n} = \sum_{i=1}^{n} \left(\frac{6^3 i^2}{n^3} + \frac{6}{n}\right) \\
&= \frac{6^3}{n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \frac{6}{n} = \frac{6^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 6 \\
&= \frac{36(n+1)(2n+1)}{n^2} + 6.
\end{aligned}$$

(In (ii) of the summation notation identities, c is constant if it does not involve the index i. This is what allows us to slip n past the summation sign.)

Putting $n = 3$, we get $A_3 = 36(4)(7)/9 + 6 = 118$, and putting $n = 6$, we get $A_6 = 36(7)(13)/36 + 6 = 97$. Both of these are in agreement with what we found in Example 36.1.1 using right endpoints (which is also what we have used in our formula for A_n).

(b) The exact area A is the limit of A_n as n goes to infinity:

$$\begin{aligned} A &= \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[\frac{36(n+1)(2n+1)}{n^2} + 6\right] \\ &= \lim_{n\to\infty}\left[\frac{72n^2 + 108n + 36}{n^2} + 6\right] = \lim_{n\to\infty}\left[72 + \frac{108}{n} + \frac{36}{n^2} + 6\right] \\ &= 78. \end{aligned}$$

(This is a reasonable answer for the exact area since it falls between the overestimates we got using right endpoints and the underestimates we got using left endpoints in Example 36.1.1.)

□

In the preceding example, we found the exact area under the curve by taking the limit of the sum of rectangle areas as the number of rectangles grew ever larger. This limit is given a name and a notation:

DEFINITE INTEGRAL. Let f be a function defined on an interval $[a, b]$. The **definite integral** of f on $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n,$$

where

$$A_n = \sum_{i=1}^{n} f(x_i)\triangle x, \quad x_i = a + i\triangle x, \quad \triangle x = (b-a)/n.$$

The symbol $\int$, called the integral sign, is meant to be a stylized S, standing for sum. So the notation for the definite integral suggests the summing of rectangle areas of height $f(x)$ and width dx as x goes from a to b, or rather the limit of such a sum as the number of rectangles goes to infinity.

The reason for the similarity between this notation and that used for the indefinite integral of f ($\int f(x)\,dx$ = most general antiderivative of f) is due to a surprising relationship between these two concepts given by the fundamental theorem of calculus (see 37).

36.2.2 Example

(a) Use the definition of the definite integral to show that

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

(b) Use elementary geometry to verify the formula of part (a).

Solution

(a) We first find A_n. The function f is given by the integrand; it is $f(x) = x$. We have

$$\begin{aligned} A_n &= \sum_{i=1}^{n} f(x_i)\triangle x = \sum_{i=1}^{n} x_i \triangle x = \sum_{i=1}^{n}(a + i\triangle x)\triangle x \\ &= \sum_{i=1}^{n}\left(a\triangle x + i(\triangle x)^2\right) = \sum_{i=1}^{n} a\triangle x + (\triangle x)^2 \sum_{i=1}^{n} i \\ &= na\triangle x + (\triangle x)^2 \cdot \frac{n(n+1)}{2} = na \cdot \frac{b-a}{n} + \left(\frac{b-a}{n}\right)^2 \cdot \frac{n(n+1)}{2} \\ &= ab - a^2 + (b-a)^2 \cdot \frac{n+1}{2n}. \end{aligned}$$

Now,

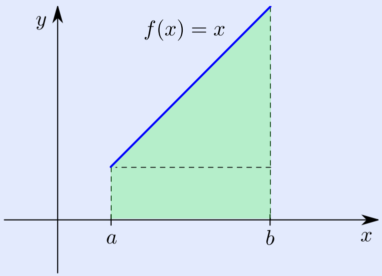
$$\lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2},$$

so

$$\begin{aligned} \int_a^b x\,dx = \lim_{n\to\infty} A_n &= \lim_{n\to\infty}\left(ab - a^2 + (b-a)^2 \cdot \frac{n+1}{2n}\right) \\ &= ab - a^2 + \tfrac{1}{2}(b-a)^2 = ab - a^2 + \tfrac{1}{2}\left(b^2 - 2ab + a^2\right) \\ &= \frac{b^2}{2} - \frac{a^2}{2}, \end{aligned}$$

as claimed.

(b) Here is the region that the definite integral gives the area of:



Breaking the region up into a rectangle and a triangle as shown and using formulas from elementary geometry, we get

$$\begin{aligned}\int_a^b x\,dx &= \text{area of rectangle } + \text{ area of triangle}\\ &= (b-a)a + \tfrac{1}{2}(b-a)^2\\ &= \frac{b^2}{2} - \frac{a^2}{2}\end{aligned}$$

(the last equation by expanding and collecting terms just as in part (a)).

□

36.3. Properties of definite integral

PROPERTIES OF DEFINITE INTEGRAL.

(i) $\displaystyle\int_a^a f(x)\,dx = 0$

(ii) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(iv) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad (c,\ \text{constant})$

(v) $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

These properties are easily verified using the definition of the definite integral. Two of the properties have geometrical interpretations that make them easy to remember:

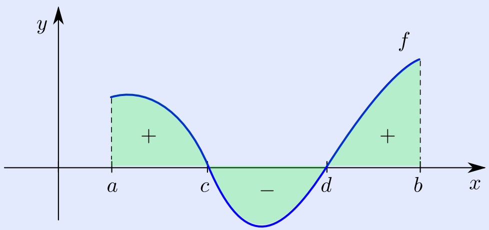
- In property (i), the left-hand side represents the area under the graph of f as x goes from a to a, so it equals zero.
- Property (iii) says that the area under the graph of f as x goes from a to b is the same as the sum of the area from a to c and the area from c to b.

36.4. Signed area

We have said that the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n$$

gives the area between the graph of f and the x-axis between $x = a$ and $x = b$. This is not quite right. In fact, the definite integral gives what is called **signed area**. It counts as positive the area of any region above the x-axis and as negative the area of any region that is below the x-axis:



The reason why areas of regions below the x-axis get counted as negative is that in the formula for A_n, the term $f(x_i)\triangle x$, which we wrote to represent the area of the ith rectangle, is negative if the graph of f is below the x-axis when $x = x_i$ (for then, $f(x_i)$ is negative).

The *actual* area of the region between the graph of f and the x-axis from a to b can be found by locating where f is zero (here, at c and d) and summing absolute values of definite integrals over the resulting intervals:

$$\text{actual area } = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^d f(x)\,dx\right| + \left|\int_d^b f(x)\,dx\right|.$$

36.4.1 Example

(a) Find $\int_{-1}^{2} x\,dx$ and interpret the results as giving signed area.

(b) Use definite integrals to express the *actual* area between the graph of $f(x) = x$ and the x-axis from $x = -1$ to $x = 2$.

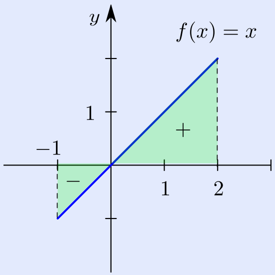
Solution

(a) Using the formula we obtained in Example 36.2.2, we get

$$\int_{-1}^{2} x\,dx = \frac{2^2}{2} - \frac{(-1)^2}{2} = \frac{3}{2}.$$

The region is composed of a triangle below the x-axis of area $\frac{1}{2}$ and a triangle above the x-axis of area 2, so

$$\text{total signed area } = -\frac{1}{2} + 2 = \frac{3}{2}.$$



(b) We have

$$\text{actual area } = \left|\int_{-1}^{0} x\,dx\right| + \left|\int_{0}^{2} x\,dx\right| = \left|-\frac{1}{2}\right| + |2| = \frac{5}{2}.$$

□

36 – Exercises

36 – 1 Approximate the area A of the region between the graph of $f(x) = 16 - (x-1)^2$ and the x-axis and between $x = 0$ and $x = 4$ as follows:

(a) Use four rectangles with the height of each rectangle given by the height of the graph above the right endpoint of the rectangle's base. Do the same using left endpoints.

(b) Repeat part (a) using eight rectangles.

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(a) Find a formula for A_n using $f(x) = 16 - (x-1)^2$ and the interval $[0, 4]$ and use it to find the sum of four rectangle areas as well as the sum of eight rectangle areas.

(b) Find the exact area A of the region between the graph of $f(x) = 16 - (x-1)^2$ and the x-axis from $x = 0$ to $x = 4$ by taking the limit of the formula you obtained in part (a).