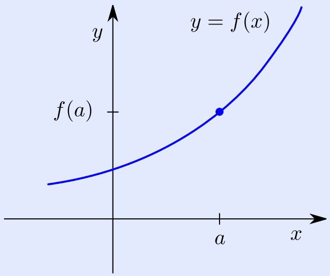
11. Continuity

11.1. Continuity at a number

A function f is said to be continuous at a number a in its domain if, roughly speaking, the graph of f can be drawn through the point $(a, f(a))$ without picking up the pencil:



f continuous at a

The notion of a limit allows us to make this idea precise:

CONTINUITY AT A NUMBER. Let f be a function and let a be in the domain of f. We say that f is **continuous at** a if $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = f(a).$$

If f is not continuous at a, it is **discontinuous at** a.

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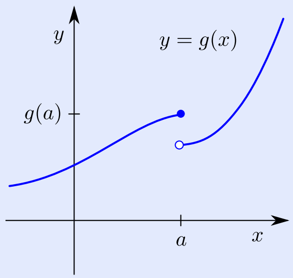
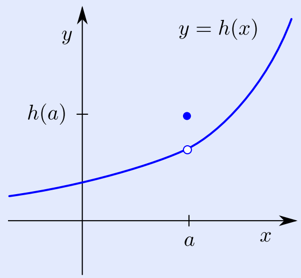
Looking at the graphs pictured below, we see that the function g is discontinuous at a due to the fact that $\lim_{x\to a} g(x)$ does not exist (the one-sided limits are different), and the function h is discontinuous at a due to the fact that $\lim_{x\to a} h(x) \neq h(a)$ (the limit is the height of the hole, while $h(a)$ is the height of the dot).



g discontinuous at a



h discontinuous at a

11.1.1 Example Decide whether the function

$$f(x) = \begin{cases} \dfrac{x^2 - 4x + 4}{x - 2}, & x \neq 2, \\ 4, & x = 2 \end{cases}$$

is continuous at 2.

Solution We have

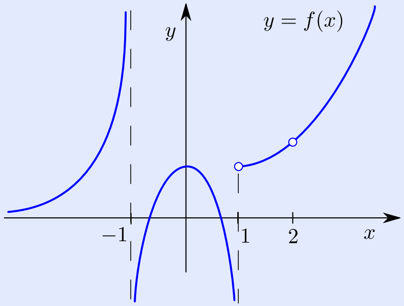
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4x+4}{x-2} = \lim_{x\to 2} \frac{(x-2)^2}{x-2}$$
$$= \lim_{x\to 2} x-2 = 0.$$

where we have used that the limit depends only on the value $f(x)$ of the function for $x \neq 2$ in the first step, the factor and cancel method, and then the substitution rule ($x-2$ is a polynomial). On the other hand, $f(2) = 4$. Therefore, $\lim_{x\to 2} f(x) = 0 \neq 4 = f(2)$ and we conclude that f is discontinuous at 2. □

11.2. Continuous function

> Continuous function. A function f is **continuous** if it is continuous at each number in its domain.

11.2.1 Example Decide whether the function f with graph as pictured is continuous.



Solution For each number a in the domain of f, if we zoom in, the portion of the graph surrounding the point $(a, f(a))$ looks roughly like the first graph of this section, that is, the graph can be drawn through this point without picking up the pencil. More precisely, for each number a in the domain of f, we have $\lim_{x\to a} f(x) = f(a)$ so that f is continuous at a. Since f is continuous at each number in its domain, it is continuous. (Although there are jumps or breaks in the graph at the numbers -1, 1, and 2, the function is not discontinuous at these numbers since they are not in the domain of f.) □

According to the definition, to say that a function f is continuous is to say that, for any number a for which $f(a)$ is defined, we can compute the limit $\lim_{x\to a} f(x)$ by just computing $f(a)$, that is, by just substituting a for x. In 6.1, it was stated that the familiar functions (polynomials, roots, exponentials, and so on), and functions built up from these, have this substitution property, so this amounts to saying that all such functions are continuous.

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11.3. Intermediate value theorem

For later use, we state a basic theorem about continuous functions. It says roughly that the graph of a continuous function on a closed interval cannot go from one height to another without attaining every height in-between. This certainly seems reasonable given our concept of such a graph as being one that can be drawn without picking up the pencil. The careful proof (omitted) depends on a property of the real numbers called the completeness axiom.

> **Intermediate value theorem.** Let f be a continuous function on a closed interval $[a, b]$. If Y is any number between $f(a)$ and $f(b)$, then there exists a number c between a and b such that $f(c) = Y$.

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11 – Exercises

11 – 1 Decide whether the function

$$f(x) = \begin{cases} \dfrac{4-x}{2-\sqrt{x}}, & x \neq 4, \\ 5, & x = 4 \end{cases}$$

is continuous at 4.

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