21. Chain rule

21.1. Statement

The power rule says that

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

This rule is valid for any power n, but not for any base other than the simple input variable x. For instance,

$$\frac{d}{dx}\left[(2x+5)^3\right] \neq 3(2x+5)^2.$$

The function $h(x) = (2x+5)^3$ is built up of the two simpler functions $g(x) = 2x+5$ and $f(x) = x^3$:

$$h(x) = (2x+5)^3 = (g(x))^3 = f(g(x)).$$

Technically speaking, h is the composition of f and g. The next rule expresses the derivative of such a function in terms of the derivatives of its components.

Chain rule. For functions f and g

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x).$$

In the composition $f(g(x))$, we call f the outside function and g the inside function. With this terminology, the rule says that the derivative of the composition of two functions is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

21.2. Examples

21.2.1 Example Find the derivative $\dfrac{d}{dx}\left[(2x+5)^3\right]$.

Solution We begin by viewing $(2x+5)^3$ as a composition of functions and identifying the outside function f and the inside function g. The outside function is the last thing you do when computing the expression for a given input x. Here, the outside function is the cubing function:

$$(2x+5)^3 = f(g(x)), \quad \text{where} \quad f(x) = x^3 \quad \text{and} \quad g(x) = 2x+5.$$

Next, we do the computations required for the chain rule formula:

$$\begin{aligned} f(x) &= x^3 & g(x) &= 2x+5 \\ f'(x) &= 3x^2 & g'(x) &= 2 \\ f'(g(x)) &= 3(2x+5)^2 \end{aligned}$$

Finally, we use the formula:

$$\begin{array}{ccccccc} \dfrac{d}{dx} & [f(g(x))] & = & f'(g(x)) & \cdot & g'(x) \\ & \downarrow & & \downarrow & & \downarrow \\ \dfrac{d}{dx} & \left[(2x+5)^3\right] & = & 3(2x+5)^2 & \cdot & 2 \end{array}$$

□

21.2.2 Example Find the derivative $\dfrac{d}{dx}\left[\sin(x^5)\right]$.

Solution Here, the outside function is the sine function:

$$\sin(x^5) = f(g(x)), \quad \text{where} \quad f(x) = \sin x \quad \text{and} \quad g(x) = x^5.$$

So

$$\begin{aligned} f(x) &= \sin x & g(x) &= x^5 \\ f'(x) &= \cos x & g'(x) &= 5x^4 \\ f'(g(x)) &= \cos(x^5) \end{aligned}$$

giving

$$\begin{array}{ccccccc} \dfrac{d}{dx} & [f(g(x))] & = & f'(g(x)) & \cdot & g'(x) \\ & \downarrow & & \downarrow & & \downarrow \\ \dfrac{d}{dx} & \left[\sin(x^5)\right] & = & \cos(x^5) & \cdot & 5x^4 \end{array}$$

□

21.2.3 Example Find the derivative $\dfrac{d}{dx}\left[\sin^5 x\right]$.

Solution Recalling that $\sin^5 x$ means $(\sin x)^5$, we see that the outside function is the one that raises an input to the fifth power:

$$\sin^5 x = f(g(x)), \quad \text{where} \quad f(x) = x^5 \quad \text{and} \quad g(x) = \sin x.$$

In order to reduce the number of steps, we go immediately to the chain rule formula and do the intermediate computations mentally as required:

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$$\begin{array}{ccccccc} \dfrac{d}{dx} & [f(g(x))] & = & f'(g(x)) & \cdot & g'(x) \\ & \downarrow & & \downarrow & & \downarrow \\ \dfrac{d}{dx} & [\sin^5 x] & = & 5(\sin x)^4 & \cdot & \cos x \end{array}$$

□

21.2.4 Example Find the derivative $\dfrac{d}{dx}\left[5^{x^2-4x+3}\right]$.

Solution Here, the outside function is the exponential function with base 5:

$$5^{x^2-4x+3} = f(g(x)), \quad \text{where} \quad f(x) = 5^x \quad \text{and} \quad g(x) = x^2 - 4x + 3.$$

Trimming the number of steps a bit more, we omit the formula for the chain rule and just think "Derivative of outside function, evaluated at inside function, times derivative of inside function":

$$\frac{d}{dx}\left[5^{x^2-4x+3}\right] = 5^{x^2-4x+3} \ln 5 \cdot (2x - 4).$$

□

21.2.5 Example Find the derivative $\dfrac{d}{dx}\left[\sqrt{5e^x + 4x^3}\right]$.

Solution The outside function is the square root function:

$$\frac{d}{dx}\left[\sqrt{5e^x + 4x^3}\right] = \tfrac{1}{2}(5e^x + 4x^3)^{-1/2}(5e^x + 12x^2).$$

□

21.2.6 Example Find the derivative $\frac{d}{dx}\left[\cos\left(e^{x^4}\right)\right]$.

Solution The outside function is the cosine function:

$$\begin{aligned}\frac{d}{dx}\left[\cos\left(e^{x^4}\right)\right] &= -\sin\left(e^{x^4}\right)\cdot\frac{d}{dx}\left[e^{x^4}\right]\\ &= -\sin\left(e^{x^4}\right)\cdot e^{x^4}(4x^3).\end{aligned}$$

The second step required another use of the chain rule (with outside function the exponential function). □

21.2.7 Example Find the derivative of $f(x) = e^{e^{e^x}}$.

Solution The chain rule is used twice, each time with outside function the exponential function:

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[e^{e^{e^x}}\right]\\ &= e^{e^{e^x}}\cdot\frac{d}{dx}\left[e^{e^x}\right]\\ &= e^{e^{e^x}}\cdot e^{e^x}\cdot e^x.\end{aligned}$$

□

21.2.8 Example Find the derivative of $f(x) = \sin\left(\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right)$.

Solution We have

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\sin\left(\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right)\right] \\
&= \cos\left(\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right)\cdot\frac{d}{dx}\left[\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right] \\
&= \cos\left(\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right)\cdot\frac{\sqrt{5x-7^{4x}}\,\frac{d}{dx}\left[x^3+2\right]-(x^3+2)\frac{d}{dx}\left[\sqrt{5x-7^{4x}}\right]}{(\sqrt{5x-7^{4x}})^2} \\
&= \cos\left(\frac{x^3+2}{\sqrt{5x-7^{4x}}}\right) \\
&\qquad\cdot\frac{\sqrt{5x-7^{4x}}(3x^2)-(x^3+2)\frac{1}{2}(5x-7^{4x})^{-1/2}(5-7^{4x}(\ln 7)(4))}{5x-7^{4x}}.
\end{aligned}
$$

□

21 – Exercises

21 – 1 Use the chain rule to find the following derivatives. Present your solution just like the solution in Example 21.2.1 (i.e., write the given function as a composition of two functions f and g, compute the quantities required on the right-hand side of the chain rule formula, and finally show the chain rule being applied to get the answer).

(a) $\frac{d}{dx}\left[(2x^4 - 5x)^7\right]$.

(b) $\frac{d}{dt}\left[\cos(8t - 3)\right]$.

21 – 2 Find the derivatives of the following functions. (You may omit details and proceed as in Example 21.2.5.)

(a) $f(x) = \sqrt[3]{8 - x^3}$.

(b) $f(t) = e^{\sin 2t}$.

21 – 3 Find the derivative of $f(x) = 2^{x^3+4x}\cos(3x)$.

21 – 4 Find the derivative of $f(t) = \cos\left(\frac{e^{5t}}{t - t^8}\right)$.